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Iteration Procedures for the Dirichlet Difference Problem

Introduction. A fundamental paper in the theory of solving partial differential equations by iteration has been written by FRANKEL.[1] In this paper Frankel discusses the RICHARDSON and LIEBMANN procedures and their corresponding accelerated procedures. The latter are termed the extrapolated Liebmann and the second-order Richardson procedures. From his paper three disadvantages of the second-order Richardson procedure as compared to the extrapolated Liebmann procedure are implied.

(1) The extrapolated Liebmann procedure appears to converge twice as fast.
(2) The extrapolated Liebmann procedure seems to require fewer operations for each step.
(3) The second-order Richardson procedure appears to require the memory of two sets of values as opposed to only one set for the extrapolated Liebmann procedure.

In the present paper all of these disadvantages will be removed for the case of the Laplace difference equation. The disadvantage of the simple Richardson procedure versus the simple Liebmann procedure will also be removed. Since the Richardson procedures do not require rapid access memory and also have simpler error eigenfunctions, the Liebmann procedures will no longer be viewed as superior (see, e.g. SHELDON & THOMAS[5]).

In addition it will be shown that knowledge of the optimum value of the parameter(s) for either accelerated procedure immediately determines the optimum value of the parameter(s) for the other accelerated procedure. These parameters will be determined if the maximum eigenvalue of the simple Richardson procedure is known, and some means based on this for estimating these parameters for regions other than rectangles will be presented. These regions will be bounded by vertical or horizontal lines connecting the equally-spaced lattice points. Curvilinear boundaries may be approximated by such boundaries.

The notation of FRANKEL[1] will be used and some familiarity with this paper will be assumed.

Some ideas presented here will be of small use if an entirely electronic computer is being used, as they will be expected to save only a few iterations. However, they will be of interest if a punched card machine is being used.

The Richardson Method. In this method the correction process applied to each ϕ^n consists of the addition of a positive multiple of $L\phi^n$ (for each interior point). Thus

$$\begin{aligned}\phi^{n+1}_{j,k} &= \phi^n_{j,k} + aL\phi^n_{j,k} \\ &\equiv \phi^n_{j,k} + \alpha[\phi^n_{j-1,k} + \phi^n_{j+1,k} + \phi^n_{j,k-1} + \phi^n_{j,k+1} - 4\phi^n_{j,k}].\end{aligned}$$

The error at each stage is

$$\epsilon^{n}_{j,k} \equiv \phi^{n}_{j,k} - \phi_{j,k}.$$

The error eigenfunctions of the operator L for a $p \times q$ rectangle are:

$$\epsilon^{(r,s)}_{j,k} = \sin(\pi r j/p) \sin(\pi s k/q) \quad \begin{cases} r = 1, 2, \cdots, p-1 \\ s = 1, 2, \cdots, q-1. \end{cases}$$

The corresponding eigenvalues are

$$L_{(r,s)} = -4[\sin^2(\pi r/2p) + \sin^2(\pi s/2q)].$$

The eigenvalues $L_{(r,s)}$ of the Laplacian operator L and the function $t \equiv \cos(\pi r/p) + \cos(\pi s/q)$, which is defined in the second of equations (27) of Frankel,[1] satisfy the relation,

(A) $$L_{(r,s)} \equiv -4 + 2t.$$

Therefore the eigenvalues of the iteration operator $K \equiv 1 + \alpha L$ are

(B) $$K_{(r,s)} \equiv 1 - 4\alpha + 2\alpha t.$$

In particular, for $\alpha = \frac{1}{4}$ (shown in Frankel[1] to be the optimum α),

(B′) $$K_{(r,s)} = t/2.$$

For convenience denote the eigenvalues of the simple Richardson operator with $\alpha = \frac{1}{4}$, for any region, by $t/2$. It will be seen that these eigenvalues are intimately connected with the eigenvalues of the other iteration processes and since they are the simplest they will be referred to as the "reference" eigenvalues.

If $\epsilon_{j,k}$ is an error eigenfunction of the simple Richardson process (formula (5) of Frankel[1]), with eigenvalue $t/2$, then $(-1)^{j+k}\epsilon_{j,k}$ is also an error eigenfunction with eigenvalue $-t/2$. Thus the non-zero reference eigenvalues occur in pairs with the same absolute value but opposite sign.

It follows that if L_0 and L_m are the smallest and largest in magnitude of the $L_{(r,s)}$, then

$$\tfrac{1}{2}t_{\max} = 1 + \tfrac{1}{4}L_0 = -[1 + \tfrac{1}{4}L_m] = -(-\tfrac{1}{2}t_{\max})$$

or

(C) $$L_0 + L_m \equiv -8.$$

See also MILNE[8], section 85.

In Frankel[1] it is shown that for $\alpha = \frac{1}{4}$ (the optimum value) the error eigenfunctions of lowest and highest frequency decrease at the slowest rate. But there is an α which removes the latter completely, still decreasing all components. This α is

(D) $$\alpha_0 = 1/(4 + 2t_{\max}).$$

Even if $t_{\max}$ is unknown, an α slightly larger than $\frac{1}{8}$ is sure to strike a heavy blow at this highest frequency error, as well as at other high frequency errors. For example, for a 20×20 square, α_0 is, to five places, 0.12577. For every eigenfunction corresponding to a $t \leq 0$, there is an α in the range $\frac{1}{8} < \alpha \leq \frac{1}{4}$ such that this removes completely that eigenfunction. [Of course, there is an α which will remove any specific eigenfunction, but unfortunately when $t \gg 0$ the α is such that the higher frequency errors (always present

due to round off) may increase tremendously.] If the problem is first solved using a coarse grid and then values assigned by interpolation, the error components will tend to be of the higher frequency type, and may be reduced in this manner. (See FLANDERS & SHORTLEY[2], p. 1330.)

In this paper[2] the lowest frequency eigenfunction has been picturesquely termed a "pillow function." This component of the error is the most troublesome to the iteration process, and the initial values should guard against it as much as possible. Thus assigning the average of the boundary values for all initial values will in general be better than assigning all zeros. It can be shown that there always exists a constant value for $\phi^0_{j,k}$ which has no pillow function error, and the average of the boundary values is a reasonable approximation to this constant. This step corresponds to a block relaxation of SOUTHWELL.[6,7]

Further block relaxations may be used (especially if the region has lines of symmetry) to reduce initially other low frequency error components.

The nearness of $\frac{1}{2}t_{\max}$ to unity for regions with a large number of interior points is an indication of the condition (see TAUSSKY[10]) of the system of linear equations. If the remaining error is of low frequency, it may seem that the difference equations are satisfied even though the approximate solution is far from the true solution. A more sensitive test is the following. It is an analogue[12] of the theorem on the vanishing of the integral of the normal derivative of a harmonic function around a closed path, and is obtained by summing the entire set of difference equations for the region. The result is that the sum of the values at the boundary points must equal the sum of the values at the interior points adjacent to the boundary, with the exception that at corners some values are counted twice and some not at all, in an obvious manner. The points are merely paired at opposite ends of segments of grid lines, and those having no mate are omitted. This principle must also be satisfied for all paths interior to the region, and may be used as a check for any low frequency error.

Even if one desires to use the other methods described in Frankel,[1] the usefulness of the simple Richardson method is not at an end. The optimum α and the optimum α and β in these methods will be shown to be determined by the reference eigenvalue of the pillow function. This eigenvalue is $\frac{1}{2}t_{\max}$.

FLANDERS & SHORTLEY[2] give a method for determining this eigenvalue experimentally. One assumes zero boundary values and approximates a pillow function. By iteration or by the non-iteration method of this paper one attempts to remove the other components present and then the rate at which the pillow function is reducing is its eigenvalue. A good approximation to the eigenvalue can then be obtained by iterating once more with $\alpha = \frac{1}{4}$ and using the Rayleigh quotient formula:

$$\frac{t_{\max}}{2} \cong \frac{\sum_{j,k} \phi^{n+1}_{j,k} \phi^n_{j,k}}{\sum_{j,k} (\phi^n_{j,k})^2} \quad \text{or} \quad \frac{t_{\max}}{2} \cong \frac{\sum_{j,k} (\phi^{n+1}_{j,k})^2}{\sum_{j,k} \phi^{n+1}_{j,k} \phi^n_{j,k}}.$$

A check on the results is presented here. If one actually has the pillow function, multiplying it by $(-1)^{j+k}$ (changing the sign at alternate points) will produce the eigenfunction of highest frequency, and then α_0 (formula (D)) will reduce it to zero in one iteration.

If one desires to solve a complete problem by the simple Richardson process, it may be observed that two independent computations are being carried on when using the optimum α of $\frac{1}{4}$, either of which may be omitted. This effectively doubles the convergence rate. The formula is

$$\phi_{j,k}^{n+1} = \tfrac{1}{4}[\phi_{j-1,k}^{n} + \phi_{j+1,k}^{n} + \phi_{j,k-1}^{n} + \phi_{j,k+1}^{n}].$$

Since $\phi_{j,k}^{n}$ is not used, it need not be computed. This means that $\phi_{j,k}^{0}$ may be assigned at points where j and k have, say, like parity (both even or both odd). Then $\phi_{j,k}^{1}$ is computed at the points where j and k have unlike parity. In general, $\phi_{j,k}^{n}$ is computed for even n at points where j and k have like parity, and for odd n at points where j and k have unlike parity. The method then becomes a Liebmann method in the sense that latest values are always used.

The Liebmann Method ($\alpha = \frac{1}{4}$). In the Liebmann method the new value of $\phi_{j,k}^{n+1}$ is used as soon as it is computed. Thus

$$\phi_{j,k}^{n+1} = \phi_{j,k}^{n} + \alpha[\phi_{j-1,k}^{n+1} + \phi_{j+1,k}^{n} + \phi_{j,k-1}^{n+1} + \phi_{j,k+1}^{n} - 4\phi_{j,k}^{n}].$$

With $\alpha = \frac{1}{4}$, this becomes

$$\phi_{j,k}^{n+1} = \tfrac{1}{4}[\phi_{j-1,k}^{n+1} + \phi_{j+1,k}^{n} + \phi_{j,k-1}^{n+1} + \phi_{j,k+1}^{n}].$$

The Extrapolated Liebmann Method. In Frankel[1] it is shown that the optimum α (equation above) is determined by the equation (35)

$$\alpha^2 t_{\max}^2 - 4\alpha + 1 = 0.$$

For the rectangle, $t_{\max} = \cos(\pi/p) + \cos(\pi/q)$. But for other regions, $t_{\max}$ is not in general known, except that it is twice the reference eigenvalue of the pillow error eigenfunction. This has not been proved here, but it follows as a corollary of results presented in YOUNG.[3] It is shown there that it is better to overestimate the optimum α than to underestimate it. Further discussion of the problem of obtaining the optimum α will be presented in a later section.

The Second-Order Richardson Method. The iteration equation for the second-order Richardson method is:

$$\phi_{j,k}^{n+1} = \phi_{j,k}^{n} + \alpha L\phi_{j,k}^{n} + \beta(\phi_{j,k}^{n} - \phi_{j,k}^{n-1}).$$

It appears that two lines of starting values are necessary. It is not clear what relation they must bear to one another or how this relation will affect later iterations. [If the coefficient of an error eigenfunction is A in $\phi_{j,k}^{n-1}$ and a in $\phi_{j,k}^{n}$, its coefficient in $\phi_{j,k}^{n+1}$ will be $a(1 - 4\alpha + \beta + 2\alpha t) - \beta A$. For $\beta = 4\alpha - 1$ this reduces to $2\alpha a t + (1 - 4\alpha)A$]. Actually this method assumes that the simple Richardson method has the sequence of values "going in the right direction" and therefore pushes them a little further.

Formula (45) of FRANKEL[1] has a misprint. It should read

$$K^* \cong 1 - \pi(p^{-2} + q^{-2})^{\frac{1}{2}}/\sqrt{2}.$$

In the third line below this formula the word "advantage" should be "disadvantage." This disadvantage, number (1) of the introduction, will now be removed. Also the above initial difficulty will disappear.

One of the formulas defining the optimum α and β in Frankel[1] is

$$\alpha(L_0 + L_m) = -2(1 + \beta).$$

By formula (C) this simplifies to

$$\beta = 4\alpha - 1. \tag{E}$$

The other formula in Frankel[1] is

$$\alpha L_0 = -(1 - \beta^{\frac{1}{2}})^2$$

or

$$1 + \alpha L_0 + \beta = 2\beta^{\frac{1}{2}}.$$

By formula (A)

$$L_0 = -4 - 2t_{\max}.$$

Thus

$$1 - 4\alpha - 2\alpha t_{\max} + 4\alpha - 1 = 2(4\alpha - 1)^{\frac{1}{2}}$$

or

$$\alpha^2 t_{\max} - 4\alpha + 1 = 0. \tag{F}$$

But this is the same equation as equation (35) of Frankel.[1] Thus the optimum α in a given region is the same in the second-order Richardson method as in the extrapolated Liebmann method.

Note that as $t_{\max} \to 2$, the optimum $\alpha \to \frac{1}{2}$.

Substituting $4\alpha - 1$ for β and writing out the operator L in full, the iteration process becomes:

$$\phi^{n+1}_{j,k} = \phi^n_{j,k} + \alpha[\phi^n_{j-1,k} + \phi^n_{j+1,k} + \phi^n_{j,k-1} + \phi^n_{j,k+1} - 4\phi^n_{j,k}] + (4\alpha - 1)(\phi^n_{j,k} - \phi^{n-1}_{j,k})$$

or

$$\phi^{n+1}_{j,k} = \alpha[\phi^n_{j-1,k} + \phi^n_{j+1,k} + \phi^n_{j,k-1} + \phi^n_{j,k+1}] + (1 - 4\alpha)\phi^{n-1}_{j,k}.$$

The formula has already simplified considerably. Disadvantage (2) of the introduction has disappeared. $\phi^n_{j,k}$ has dropped out. As in the simple Richardson method, it is unnecessary to compute it at all. That is, if one has values at alternate points, say where $j + k$ is odd, and values in the next iteration at points where $j + k$ is even, then one can compute the values in the succeeding iteration at points where $j + k$ is odd, and from these the values in the iteration after that at points where $j + k$ is even again. The missing values are independent of the values computed, and vice versa. Thus only half the computation is necessary. This cancels the factor of two disadvantage over the extrapolated Liebmann method.

This method of computing removes another disadvantage, number (3) of the introduction, of the second-order Richardson method for machines with limited memory. At no time is it necessary to remember more than one value at any point. The method is a Liebmann method in the sense that latest values are always used. In fact this method and the extrapolated Liebmann method are both special cases of the "successive overrelaxation method" developed by Young,[3] depending only on "ordering" of the points. This reduced second-order Richardson method is actually produced by an "alternate diagonal" ordering of Young.[3] However, there is one difference,

which can perhaps be used to advantage. In one case there are "missing" values. In the other case two successive iterations are collapsed into one. This of course changes the eigenfunctions. In the reduced second-order Richardson method, the eigenfunctions are still the simple Richardson eigenfunctions.

Use of a Richardson procedure now permits the lowest frequency error components to be attacked both by initial block relaxations and by solution first with a coarse grid. The highest frequency error component can be eliminated in one iteration by the simple Richardson procedure with $\alpha = \alpha_0$ as given by formula (D) and other high frequency components can be effectively eliminated in a like manner.

However, unlike the simple Richardson procedure, for the second order Richardson procedure all error components become equally "troublesome," at least in the simple sense of asymptotic decay rate.[11] Attacking them individually no longer gives such an obvious advantage. However, examine the expression $2\alpha at + (1 - 4\alpha)A$ (given above in brackets) for the coefficient of an error eigenfunction in terms of its coefficients in the two previous lines. Observe that one will probably obtain a second line from an initial approximation by an iteration using the simple Richardson procedure with $\alpha = \frac{1}{4}$. Then $a = At/2$ and the expression becomes $(\alpha t^2 + 1 - 4\alpha)A$. Considered as a function of t this is a parabola with minimum at $t = 0$. This suggests that one will probably gain by making some effort to eliminate the lowest and highest frequency components and then iterating at least twice by the simple Richardson procedure with $\alpha = \frac{1}{4}$ before starting the second order procedure (assuming, of course, that the computer being used is not of the highest speed).

The Reference Eigenvalue for the Pillow Function. Since the reference eigenvalue for the pillow function is $\frac{1}{2}t_{\max}$ and the optimum α in the faster converging methods is determined by the smaller root of the equation

$$\alpha^2 t_{\max}^2 - 4\alpha + 1 = 0,$$

this eigenvalue if of prime importance. A practical method for approximating its value, using the Rayleigh quotient, has already been mentioned. Unfortunately this method in general gives a value which is too small, and as mentioned previously, it is much better to have a value which is too large. Since a formula is known for the rectangle, it is useful to observe that if the region in question is covered by a rectangle, the desired value is less than the reference pillow eigenvalue of the rectangle. Similarly a rectangle interior to the region will have a smaller reference pillow eigenvalue. In making these estimates, better results can be obtained if it is realized that for a fixed number M of interior points the reference pillow eigenvalue is a measure of the compactness of the region. E.g., an unbounded number of interior points can be strung out so that the maximum reference eigenvalue is less than one-half. But only four interior points arranged in a square will have a reference eigenvalue of one-half. This means that any arms jutting out from the region can be cut off and used to fill in any part of the covering rectangle without reducing the maximum reference eigenvalue of the region in question. This also means that, unless the region is almost circular, an upper bound to the reference eigenvalue is $\cos \pi/(1 + M^{\frac{1}{2}})$, being the

maximum reference eigenvalue of a square with M interior points, and a square being a rather compact region.

Time Estimates. The tables of Frankel,[1] page 74, can now be changed to:

Method	T
Laplace-Richardson ($\alpha = \frac{1}{4}$)	4τ
Laplace-Liebmann ($\alpha = \frac{1}{4}$)	4τ
Laplace-Liebmann (optimum α)	6τ
Laplace-2nd-order Richardson	6τ.

Approximate Total Calculating Time.

	$N = 10$	$N = 20$	N Large
Laplace-Richardson ($\alpha = \frac{1}{4}$) Laplace-Liebmann ($\alpha = \frac{1}{4}$)	$5 \cdot 10^4\tau$	$8 \cdot 10^5\tau$	$5.6N^4\tau$
Laplace-Liebmann (optimum α) Laplace-2nd-order Richardson	$10^4\tau$	$10^5\tau$	$13N^2\tau$

Here τ is the mean time required for each arithmetic operation, T is the time for the evaluation of one $\phi^n_{j,k}$ and N is the common value of p and q for a square with $(N-1)^2$ interior points.

The change in the extrapolated Liebmann method (for optimum α) from 7τ to 6τ in the first table is because terms may be collected in the iteration formula to give:

$$\phi^{n+1}_{j,k} = \alpha[\phi^{n+1}_{j-1,k} + \phi^n_{j+1,k} + \phi^{n+1}_{j,k-1} + \phi^n_{j,k+1}] + (1 - 4\alpha)\phi^n_{j,k}$$

and $1 - 4\alpha$ may be computed when α is computed.

The improved second-order Richardson method now has the same rate of convergence as the extrapolated Liebmann method, requires no more memory, and does not require rapid access memory. Thus it would seem to be a superior method for general use.

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A Method for Calculating Inverse Trigonometric Functions

Introduction. The problem of calculating an angle from a known trigonometric function of that angle occurs frequently in the numerical analysis of physical systems. The purpose of this paper is to introduce a new method which is particularly suitable for the I. B. M. Card Programmed Calculator.

Methods. There are several methods of determining an angle from its trigonometric functions; the more commonly employed ones are:

1. Table look up.
2. Infinite processes
   (a) Power series
   (b) Continued fractions.[1]
3. Approximations.[2]

Which of these methods is the best to use depends primarily on the computing equipment that is to be used. For desk calculating, tables are usually best; for computers with limited storage capacity, the infinite processes are usually used; on large scale computers, the approximations are most efficient. Since the IBM-CPC has a very limited amount of high speed storage, the common practice has been to use a Maclaurin's series with a variable number of terms.[3] Other methods have been devised[4] that speed up the process by first transforming the argument into a smaller number so that the series will converge more rapidly.

The New Method. The above discussion helps to illustrate the problem facing the programmer when programming a method for determining an angle from its trigonometric function. An entirely different method for finding inverse functions will now be described. This method is more of a logical process than a mathematical one. The following notation will be used:

$$y = \text{trig}^{-1}(A, B)$$

which reads, "y is the inverse trigonometric function of A and B." Furthermore,

$$A = \sin y,$$
$$B = \cos y.$$

In the case both the sine and cosine of y may be found independently, this method will determine the angle in its proper quadrant. If the cosine must be found by an identity from the sine, then only the principal value will be computed.

Let us assume that the sine and cosine of some angle, y, is known and it is subject to the following restrictions:

$$0 < y < \pi,$$

$$y \neq \frac{\pi}{2^m} \quad (m = 0, 1, 2, \cdots, 25).$$

The excluded values are special cases and will be discussed later. We first calculate sin $2y$ and cos $2y$ by suitable trigonometric formulae. From the

first restriction, $0 < y < \pi$, we have $\sin y > 0$; if

$$\sin 2y > 0, \quad \text{then} \quad 0 < 2y < \pi \quad \text{and} \quad 0 < y < \frac{\pi}{2},$$

but if

$$\sin 2y < 0, \quad \text{then} \quad \pi < 2y < 2\pi \quad \text{and} \quad \frac{\pi}{2} < y < \pi.$$

It is obvious then that a balance test of $\sin 2y$ will determine the quadrant of y; this information may be retained by storing the initial value of the quadrant. Calling the stored quantity Σ_1, it is seen that if

$$0 < y < \frac{\pi}{2}, \quad \text{then} \quad \Sigma_1 = 0,$$

but if

$$\frac{\pi}{2} < y < \pi, \quad \text{then} \quad \Sigma_1 = \frac{\pi}{2}.$$

From $\sin 2y$ and $\cos 2y$ we compute $\sin 4y$ and $\cos 4y$ and analyze in the same manner.

$$\text{If} \quad \sin 2y > 0 \quad \text{and} \quad \sin 4y > 0, \quad \text{then} \quad 0 < y < \frac{\pi}{4} \quad \text{and} \quad \Sigma_2 = 0,$$

$$\text{if} \quad \sin 2y > 0 \quad \text{and} \quad \sin 4y < 0, \quad \text{then} \quad \frac{\pi}{4} < y < \frac{\pi}{2} \quad \text{and} \quad \Sigma_2 = \frac{\pi}{4},$$

$$\text{if} \quad \sin 2y < 0 \quad \text{and} \quad \sin 4y > 0, \quad \text{then} \quad \frac{\pi}{2} < y < \frac{3\pi}{4} \quad \text{and} \quad \Sigma_2 = \frac{\pi}{2},$$

$$\text{if} \quad \sin 2y < 0 \quad \text{and} \quad \sin 4y < 0, \quad \text{then} \quad \frac{3\pi}{4} < y < \pi \quad \text{and} \quad \Sigma_2 = \frac{3\pi}{4},$$

where Σ_2 is the initial value of the octant in which y lies. A comparison of the Σ_2's and Σ_1's will show that when

$$\sin 4y > 0, \quad \text{then} \quad \Sigma_2 = \Sigma_1$$

and when

$$\sin 4y < 0, \quad \text{then} \quad \Sigma_2 = \Sigma_1 + \frac{\pi}{4}.$$

In fact, if the same analysis is extended the following generalization will be apparent:
When

$$\sin 2^n y > 0, \quad \text{then} \quad \Sigma_n = \Sigma_{n-1},$$

and when

$$\sin 2^n y < 0, \quad \text{then} \quad \Sigma_n = \Sigma_{n-1} + \frac{\pi}{2^n}.$$

The Σ_n will be the initial value of the interval, of width $\pi/2^n$, in which y lies. If this process is continued until $\pi/2^n$ is less than the maximum allowable error, then Σ_n will be the value of y. If the error is to be less than 5×10^{-8} then n must take on all values up to 26.

The cases where $y = \pi/2^m$ cause difficulty because

$$\sin 2^n y = 0 \quad \text{if} \quad n \geqq m.$$

Actually, the number Σ_n is the answer, y, when $n = m$ since $\Sigma_n = \Sigma_m = \frac{\pi}{2^m}$. Therefore, the first zero should be treated as a "minus" and all subsequent zeroes as "plus".

If

$$y = 0 \quad \text{or} \quad y = \pi,$$

then

$$\sin y = 0$$

and

$$\cos y = 1 \quad \text{or} \quad \cos y = -1;$$

therefore, if $\sin y$ is zero, then a separate test must be made on the cosine to determine if $y = 0$ or $y = \pi$.

The discussion thus far has assumed that y is in the first two quadrants; the method can be extended to third and fourth quadrant angles by three methods:

1. The absolute value of $\sin y$ may be used and the final answer multiplied by minus one if $\sin y$ was originally negative. This gives answers in the range $-\pi < y \leqq \pi$.
2. If $\sin y$ is negative, set $\Sigma_0 = \pi$; in the previous discussion it was implied that $\Sigma_0 = 0$. This will give answers in the range $0 \leqq y < 2\pi$.
3. If $\sin y$ is negative, set $\Sigma_0 = -\pi$. This puts the same limits on y as method 1.

In most cases, either method 1 or method 3 is preferable to method 2.

Method 1 is illustrated in the flow diagram, Figure 1. In practice it is more convenient to store $\sigma_n = \Sigma_n/\pi$ and $p_n = 1/2^n$ instead of Σ_n and $\pi/2^n$, respectively. Since $p_n = \frac{1}{2}p_{n-1}$, any rounding error will be positive and the sum σ_n will include the errors of all p_k's where $k \leq n$. This error may be as large as six in the last digit. This error may be reduced by always rounding to an even number or by carrying an extra digit in p_n and σ_n; obviously both methods have disadvantages. The following is still another way in which the rounding error may be reduced and also save storage space. Let $a_n = 0$ or $a_n = 1$, depending on whether the term is to be omitted or added; also let $n = 0, 1, 2, \cdots, N$; then

$$y = \Sigma_N = \pi\sigma_N$$

and

$$\sigma_N = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots + \frac{a_N}{2^N};$$

factoring out $\frac{1}{2^N}$, then

$$\sigma_N = \frac{1}{2^N}(a_1 2^{N-1} + a_2 2^{N-2} + a_3 2^{N-3} + \cdots + a_N)$$

or

$$2^N \sigma_N = \{\{[(2a_1 + a_2)2 + a_3]2 + a_4\}2 + \cdots + a_N\}.$$

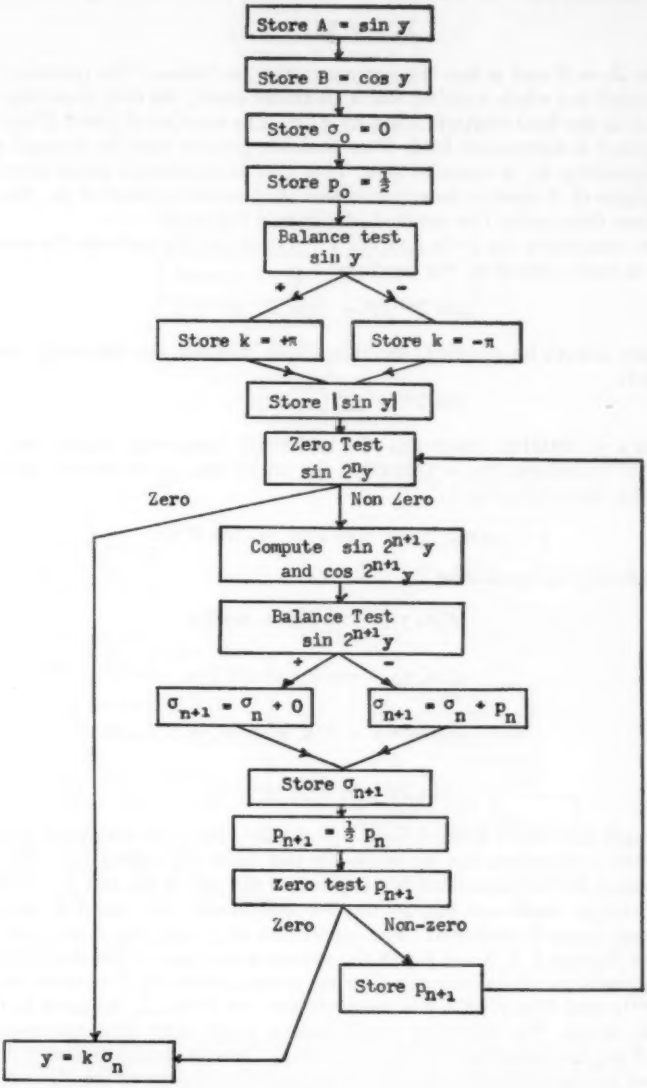
Store A = sin y
Store B = cos y
Store $\sigma_o = 0$
Store $p_o = \frac{1}{2}$
Balance test sin y
+
-
Store k = +π
Store k = -π
Store |sin y|
Zero Test sin $2^n y$
Zero
Non Zero
Compute sin $2^{n+1}y$ and cos $2^{n+1}y$
Balance Test sin $2^{n+1}y$
+
-
$\sigma_{n+1} = \sigma_n + 0$
$\sigma_{n+1} = \sigma_n + p_n$
Store σ_{n+1}
$p_{n+1} = \frac{1}{2} p_n$
Zero test p_{n+1}
Zero
Non-zero
Store p_{n+1}
$y = k\sigma_n$

FIG. 1.

The quantity $Z_N = 2^N \sigma_N$ may be computed by the following formula:

$$Z_{n+1} = 2Z_n + a_{n+1}$$

where $Z_0 = 0$ and n has the same meaning as before. The quantity thus computed is a whole number and is therefore exact, the only rounding error will be in the final multiplication by $\pi/2^N$. The number of times Z has been computed is subtracted from N so that the process may be stopped when this quantity, b_n, is equal to zero. This also saves storage space since only two digits of N need to be stored rather than seven or more of p_n. The flow diagram illustrating this method is shown in Figure 2.

In computing $\sin 2^{n+1}y$ and $\cos 2^{n+1}y$ both $\sin 2^n y$ and $\cos 2^n y$ must be used in each; otherwise the condition

$$(\sin 2^{n+1}y)^2 + (\cos 2^{n+1}y)^2 = 1$$

will not always be satisfied. As an example, suppose the following formula is used:

$$\cos 2^{n+1}y = 2(\cos 2^n y)^2 - 1.$$

If $\sin y \leq .0003162$, then $\cos y = 1.0000000$ (the error being less than 5×10^{-8}) and $\cos 2^n y = 1.0000000$ for all n; this is obviously incorrect. A better formula to use is

$$\cos 2^{n+1}y = (\cos 2^n y)^2 - (\sin 2^n y)^2.$$

By defining the quantities

$$F(n, y) = \cos 2^n y + \sin 2^n y$$

and

$$G(n, y) = \cos 2^n y - \sin 2^n y,$$

then

$$\cos 2^{n+1}y = F(n, y) \cdot G(n, y)$$

and

$$\sin 2^{n+1}y = F(n, y)^2 - 1.$$

Although the above formulations are clumsy, they are necessary in order that the calculation can be made on the IBM 605 calculator. The programming for an unmodified 605 is given at the end of the article. All three-digit storage units are put on an 8–6 assignment and one full sweep of program steps is impulsed. The quantities $\sin y$ and $\cos y$ are read into Factor Storage 1, 2, 3 and 4 and the answer is read out of the third through tenth positions of the counter. During computation, F1, 2 is used to store $\sin 2^{n+1}y$ and $F(n, y)$; F3, 4 is used to store $\cos 2^{n+1}y$; Z_n is stored in G1, 2 and b_n in G3. The following modifications would save program steps but would require additional tabulator wiring. General storage 1 and 2 can be cleared by the tabulator instead of on program step 10 to set $Z_0 = 0$. The quantity $b_0 = N$ may be read from a card or emitted from the tabulator into G3. If the tube located at panel 1–7T is removed, the absolute value of a number may be read in on one program step instead of taking three. If the final multiplication is carried out on a separate card cycle, steps 50

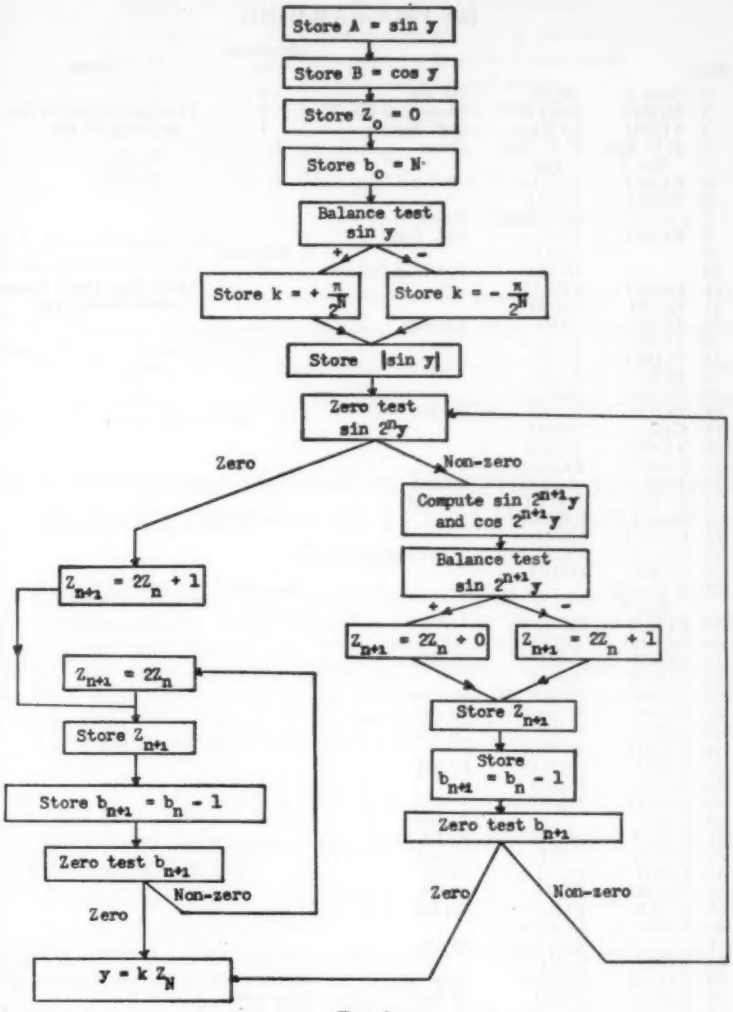

FIG. 2.

through 60 may also be eliminated. This programming allows y to be any value in the region $-\pi < y < \pi$, but excludes the possibility $y = \pi$.

In the worst cases, this method requires a little over two seconds which in CPC operation is less than six card cycles. This compares favorably with the time required by other methods to attain the same accuracy, and unlike most other methods does not require extra card cycles for determining the quadrant of the answer.

605 PROGRAMMING

Step				Suppress on	Notes
1	Emit 2	MQRI	Ri 2nd	1	
2	MQRO	Emit 6	Prog. Exp No. 1	1	Program expansion No. 1 is G3RI, Ri 6th
3	F12RO	ECRI−	Bal. Test	1	
4	P. U. Rpt. No. 1	P. U. Rpt. Del.	Reset	1, −	
5	F12RO	ECRI+		1, −	
6	ECRO	F12RI			
7		Zero Test	Reset		
8	F12RO	ECRI+	Bal. Test		
9		G3RI		1, Non zero	
10		G12RI	P. U. Gp. Sup. 1	1	
11	F34RO	ECRI+	*		* D.O. Rpt. Del. if Repeat selector one is up.
12	ECRO	MQRI			
13	ECRO	G4RI	Ro 6th		
14	F12RO	ECRI−			
15	F12RO	ECRI−			
16	ECRR	F12RI			
17	F12RO	Mult+			
18	ECRR	F34RI	Ro 6th		
19	G4RO	MQRI			
20	F12RO	Mult+			
21	F34RO	ECRI+			
22	½ adj.		Ri 2nd		
23	ECRR	F34 RI	RO 3rd		
24	Emit 1	ECRI+		−, 2	
25	G12RO	ECRI+			
26	G12RO	ECRI+			
27	ECRR	G12RI			
28	P. U. Gp. Sup 2			Non zero	
29	F12RO	ECRI+			
30	ECRO	MQRI			
31	ECRR	G4RI	Ro 6th		
32	F12RO	Mult+			
33	ECRR	F12RI	Ro 6th		
34	G3RO	ECRI+	Ri 6th		
35	G4RO	G3RI	Ri 6th		
36	F12RO	ECRI+			
37	G3RO	MQRO	F12RI		
38	G4RO	MQRI			
39	F12RO	Mult+			
40	½ adj.		Ri 2nd		
41	Emit 1	MQRI	Ri 5th		
42	ECRO	G3RI	Ro 6th		
43	G3RO	ECRI−	Ri 6th		
44	MQRO	ECRI−	Ri 6th		
45	ECRR	F12RI	Ro 3rd		
46	G3RO	ECRI+			
47		Zero Test	Ri 6th		
48	ECRR	G3RI			
49	F12RO	ECRI−	Prog. Rpt.	Zero	
50	Emit 4	F34RI	Ri 2nd	Non zero	
51	Emit 8	Prog. Exp 2	Ri 2nd	Non zero	Program expansion 2 is ECRI+, Ri 6th
52	Emit 3	ECRI+	Prog. Exp 3	Non zero	Program expansion 3 is Ri 3rd, Ri 4th
53	Emit 1	ECRI+	Ri 5th	Non zero	
54	ECRR	MQRI	Ro 2nd	Non zero	
55	G12RO	*		Non zero	* Multiply plus if repeat selector 1 is normal, minus if transferred
56	F34RO	MQRI	Emit 6	Non zero	
57	ECRR	F34RI	Ro 6th	Non zero	
58	F34RO	ECRI+		Non zero	
59	G12RO	*		Non zero	* Multiply plus if repeat selector 1 is normal, minus if transferred
60	½ adj.		Ri 2nd	Non zero	

Steps	Description of Operation
8, 11–13	Compute $\cos x - \sin x$ and store
14–16	Compute $\cos x + \sin x$ and store
17–23	Compute $\cos 2x$ and store
24–27	Compute Z_{n+1} and store
29–45	Compute $\sin 2x$ and store
46–48	Modify b_n
50–60	Multiply Z_N by $\pi/2^N$

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[1] D. TEICHROEW, "Use of continued fractions in high speed computing," *MTAC*, v. 6, p. 127–133.
[2] C. HASTINGS, *Approximations in Numerical Analysis*. (Form 15's). Rand Corporation, Santa Monica. Sheets 8–12, 35–37, 39.
[3] W. P. HEISING, "An eight-digit general-purpose control panel," IBM Tech. News Letter No. 3, 1951.
[4] D. W. SWEENEY, "A Model II four address floating-decimal coding system," IBM Tech. News Letter No. 5, 1953.

A Partitioning Method of Inverting Symmetric Definite Matrices on a Card-Programmed Calculator

Although certain standard lines of approach seem to exist in all methods of inverting matrices, the problem of inversion has not yet been solved with finality, and modifications on "well-worn" solutions often prove to be of value. This paper discusses a partitioning method for inverting matrices which is particularly suited to the International Business Machines' Card Programmed Calculator, (C.P.C.), and subsidiary equipment.

The general problem is to invert an n^{th} order matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a and d are each square submatrices of order n_1 and $n_2[n_1 + n_2 = n]$. Let the inverse of M be

$$M^{-1} = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$$

partitioned in the same manner as the original matrix. Post-multiplying M by its inverse, we have

$$\begin{pmatrix} aA + bB & aC + bD \\ cA + dB & cC + dD \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or in equation form

$$\begin{aligned} aA + bB &= 1 \quad \text{(identity matrix of order } n_1\text{)} \\ cA + dB &= 0 \quad \text{(zero matrix } n_2 \times n_1\text{)} \\ aC + bD &= 0 \quad \text{(zero matrix } n_1 \times n_2\text{)} \\ cC + dD &= 1 \quad \text{(identity matrix of order } n_2\text{)}. \end{aligned}$$

Solutions of these equations are

$$
\begin{aligned}
D &= (d - ca^{-1}b)^{-1} \\
C &= -a^{-1}bD \\
B &= -Dca^{-1} \\
A &= a^{-1} - a^{-1}bB.
\end{aligned} \tag{1}
$$

Thus to find the inverse M^{-1}, the procedure is to determine a^{-1} from which we can derive D, C, B, and finally A. If the original matrix is symmetric, equations (1) simplify to

$$
\begin{aligned}
D &= (d - b'a^{-1}b)^{-1} \\
B' &= C = -a^{-1}bD \\
A &= a^{-1} - a^{-1}bB.
\end{aligned} \tag{2}
$$

The restriction to definite matrices M assures us that $(d - b'a^{-1}) \neq 0$.

Let b and c be a column and row vector, respectively, and d a scalar; then the solution of (2) is obtained relatively simply once a^{-1} is known. (Note that D is merely the reciprocal of a scalar.) a^{-1} can be found by partitioning a in an analogous manner and a^{-1}, which is of the order $n - 1$, depends on the inverse of an $n - 2$ submatrix. The process carried to its completion implies that the inverse of the original matrix M is a function of the inverses of smaller and smaller submatrices. Thus a well known method for inverting matrices is suggested: start by inverting a small submatrix, say of order 2; from the 2×2 inverse, derive the 3×3 inverse, and continue until the desired $n \times n$ inverse is computed. This method turns out to be efficient and very simple to handle on an automatic computer.

The programming suggested assumes the following C.P.C. set-up: (a) Products can be accumulated in the 605 Electronic Calculating Unit counter; e.g., if $\alpha = x_1y_1$ is derived on card p and $\beta = x_2y_2$ is derived on card $p + 1$, then on card $p + 1$, $\alpha + \beta$ can be accumulated in the counter. In other words, the C.P.C. must be able to perform both a multiplication and an addition on one card.[1] (b) A second required feature is that a card may be read from either a normal or an alternate instruction field.

The procedure employs $(2n - 1)$ storage locations.

The manual operating scheme is:

1. Load the 527 Summary Punch with cards pre-punched in their instruction fields. There are n stages and at the r^{th} stage, the 527 summary punches r^2 cards which are fed back into the 418 Tabulator at stage $(r + 1)$; the r^2 cards are the r^{th} order inverse a^{-1}.

2. At the beginning of the r^{th} stage, the operator removes $(r - 1)^2$ cards summary punched on the $(r - 1)^{\text{th}}$ stage and runs them through the collator which assembles them in row-by-row sequence. These $(r - 1)^2$ cards are manually put in the program deck for stage r. The first part of this deck is run through the tabulator; when the $(r - 1)^2$ cards have gone through the tabulator once, they are removed and placed at the end of the program deck for the r^{th} stage and the deck finishes its run. A new set of r^2 cards will have been summary punched and the operator is ready to go on to stage $(r + 1)$.

The table below gives the functional computing procedure at the r^{th} stage. The following notation is used: x is a storage bank having n locations $[x+1, \cdots, x+n]$ and y a storage bank having $n-1$ locations, $[y+1, \cdots, y+(n-1)]$.

Table 1. Symmetric Matrix—Stage r

Step	Function	Put into Storage Location	Operation	Number of Cards
1	Load b	$x+1$ ⋮ $x+(r-1)$	(Add)	$r-1$
	Load d	$x+r$	(Add)	1
2	Form $a^{-1}b$	$y+1$ ⋮ $y+(r-1)$	Row by row, the $(r-1)^2$ cards containing a^{-1} are put through the tab. The first instruction field is punched so as to give the product of each element of a row of a^{-1}, from the card, times the corresponding element of b, from storage x, accumulate these products for each row and store the $(r-1)$ sums in y.	$(r-1)^2$
3	Form $(d - b'a^{-1}b)$	Keep in 605 Counter	Call out the element of b from storage x, negative multiply by the elements of $a^{-1}b$ in storage y, and accumulate the products with d which was called in on the first card from $x+r$.	$r-1$
	Take $\frac{1}{(d - b'a^{-1}b)} = D$	$x+r$	Division	
4	Calculate $-a^{-1}bD = C$	Summary Punch	Call out each $a^{-1}b$ from storage y, negative multiply by D and punch out.	$r-1$
5	Calculate $-a^{-1}bD = B$	Summary Punch and Store: $x+1$ ⋮ $x+(r-1)$	Same as 4.	$r-1$
6	D	Summary Punch	Take D from storage $x+r$ and summary punch.	1
7	Calculate $a^{-1} - a^{-1}bB = A$	Summary Punch	Run the $(r-1)^2$ cards through the tab again, this time using second instruction field. As each row of a^{-1} goes through, take the element of $a^{-1}b$ from storage y, and negative multiply it by the element of B in storage x, add it to the element of a^{-1} from the card-summary punch the answer.	$(r-1)^2$
			Total number of cards on the r^{th} stage.	$2r^2+1$

Notice the order of punching out: we first obtain the last element in each row of a^{-1} for the first $(r-1)$ rows (C); then we obtain the entire r^{th} row (B, D); and finally the first $(r-1)$ rows less their last element (A). If we break this deck into a deck of C, B, and D (already in order) and a deck of A

Table 2. Approximate C.P.C. Running Time for Symmetric Matrices

Order n	10	16	20	25	30	35	40
number of cards	816	3112	5928	11,344	19,332	30,392	45,024
running time	14 min.	52 min.	1 hr. 35 min.	3 hrs. 10 min.	5 hrs. 20 min.	8 hrs. 20 min.	12 hrs. 30 min.

we can put one deck into each feed of the collator and merge them into proper row-by-row order.

The approximate total number of cards[2] passing through the C.P.C. is $\frac{2}{3}n^3 + \frac{1}{2}n^2$; the *average* speed of the C.P.C. for this method is 60 cards/minute.

If a high speed automatic calculator is used where storage is less limited, then the major time consuming factor, summary punching, can be eliminated and the relative computing time shortened.

In addition to the speed of the method, the procedure has several other recommending features: if inverses of submatrices are needed, these are available by-products of the calculations. Similarly the programming of an $n \times n$ inverse includes the programming of the inverses of smaller matrices.

An example of a 5×5 matrix inverted by this procedure is

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 8 & 11 & 14 \\ 3 & 8 & 14 & 20 & 26 \\ 4 & 11 & 20 & 30 & 40 \\ 5 & 14 & 26 & 40 & 55 \end{bmatrix} \quad M^{-1} = \begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}.$$

The procedure gave the exact inverse in less than .75 of a minute.

Under the assumption that in general it takes n^3 operations to invert an n^{th} order matrix, H. HOTELLING[3] asserts, "In partitioning a matrix [for inversion], there is an advantage in dividing the rows into *equally numerous* groups, since when the sum of two positive numbers is fixed, the sum of their cubes is a minimum when they are equal." If Hotelling's inference were correct, the method presented in this article of partitioning by single rows and columns would be inefficient. But in actuality the number of calculations required to invert any matrix by partitioning is invariant under the mode of partitioning:

With respect to computing time, it is most efficient to use single row and column partitioning since this method enables us to store b and d, and also at later steps $a^{-1}b$, D, and B. If b were not a vector and d not a scalar, then storage of these quantities would become impractical and the speed of

Table 3

Function	Approximate Number of Operations
a^{-1}	n_1^3
$a^{-1}b$	$n_1^2 n_2$
ca^{-1}	$n_1^2 n_2$
$ca^{-1}b$	$n_1 n_2^2$
$(d - ca^{-1}b)^{-1}$	n_2^3
$-a^{-1}bD$	$n_1 n_2^2$
$-Dca^{-1}$	$n_1 n_2^2$
$a^{-1} - a^{-1}bB$	$n_1^2 n_2$
Total	$n_1^3 + 3n_1^2 n_2 + 3n_1 n_2^2 + n_2^3 = (n_1 + n_2)^3 = n^3$

operation would be slowed down further by more intermediate summary punch operations.

It can be shown that if n_0 is the order of the largest matrix which can be inverted given the storage capacity, and $n_0 < n$, then the most efficient method of inverting by partitioning requires inverting of, say, k submatrices, where the first $(k-1)$ matrices inverted are of order n_0, and the k^{th} inverted matrix is of order $n - (k-1)n_0$.

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This article is contained in H. M. WAGNER, "Matrix Inversion on an Automatic Calculator by Row and Column Partitioning," P-417, The RAND Corporation, July 14, 1953; P-417 also discusses in detail (1) the inversion by partitioning of non-symmetric matrices; (2) optimal partitioning; (3) inversion of n^{th} order matrices where n exceeds storage limitations.

[1] W. ORCHARD-HAYS, "The Duplex System for IBM's Model II CPC, A Fast Four Address, Double Operation, Floating-Decimal Set-Up," RM-1044, The RAND Corporation, February 23, 1953; H. M. WAGNER, "Coder's Manual for Duplex System" and "Stanford's Revised Duplex System—Wiring Manual," Technical Reports 1 and 4, Stanford Computation Center. These references give set-ups which meet the two operations per card requirement.

[2] The formula and table following do not take into account any saving in cards and time at the beginning of the procedure when r is small enough to invert without any summary punching. E.g., with 19 storage locations, a 5×5 matrix can be inverted without any summary punching in 90 cards at 150 cards/minute. In general, if S is the number of storage locations available, then all n^{th} order symmetric matrices for which $n \leq N$ where $\frac{N^2 + 3N - 2}{2} = S$ can be inverted without any summary punching.

[3] H. HOTELLING, "Some computational devices," Chapter X in *Statistical Inference in Dynamic Economic Models*, Cowles Commission Monograph 10, T. C. KOOPMANS, ed. New York, 1950, p. 323–328.

Polynomial Approximations to Elementary Functions

The increasing use of high-speed computing machines has revived interest in the approximation of functions of a real variable, particularly by polynomials. An orthodox table of function values at equidistant arguments may require considerable storage space in an electronic machine. In contrast, the coefficients of a polynomial which represents the function to a desired accuracy over a specified range may require very little storage space, and simple instructions will suffice for the evaluation of the polynomial.

If a function is bounded and continuous in a given finite range of argument, a powerful polynomial approximation is usually obtained by truncation of the infinite expansion of the function in Chebyshev polynomials. Many properties of these polynomials, and the means by which the coefficients in the infinite expansions can be derived, are given by LANCZOS,[1] whose notation for the Chebyshev polynomials in the range $0 \leq x \leq 1$ will be used here

$$T_n^*(x) = \cos n\theta, \quad \cos\theta = 2x - 1.$$

In this note, tables are given which ease the calculation of polynomial approximations to some common functions. Coefficients in the infinite

Chebyshev expansions valid in the specified ranges are given to nine decimal places, and formulae are also given for the Chebyshev polynomials.

A polynomial approximation to any of the functions considered is obtained by truncation of the infinite Chebyshev series. The retained coefficients are then rounded-off, and the polynomial rearranged in powers of the independent variable.

As an example, an approximation to $\sin \frac{1}{2}\pi x$ in the range $-1 \leq x \leq 1$ is

$$x \sum_{n=0}^{5} A_n T_n^*(x^2),$$

where the coefficients are given in the tables. Substituting their values gives

$$\begin{aligned}
\sin \tfrac{1}{2}\pi x \doteq\ & x\{1.27627\ 8962 - 0.28526\ 1569(2x^2 - 1) \\
& + 0.00911\ 8016(8x^4 - 8x^2 + 1) \\
& - 0.00013\ 6587(32x^6 - 48x^4 + 18x^2 - 1) \\
& + 0.00000\ 1185(128x^8 - 256x^6 + 160x^4 - 32x^2 + 1) \\
& - 0.00000\ 0007(512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1)\} \\
=\ & 1.57079\ 6326x - 0.64596\ 4102x^3 \\
& + 0.07969\ 2704x^5 - 0.00468\ 1984x^7 \\
& + 0.00016\ 0640x^9 - 0.00000\ 3584x^{11}
\end{aligned}$$

In this case, the rapidity of convergence of the Chebyshev series indicates that the truncation error will be small, and most of the error in the above approximation arises from the rounding-off of the coefficients A_n. Thus the maximum error cannot exceed three units in the ninth decimal place.

If an approximation is required to less than nine decimal accuracy, unwanted rounding errors can be avoided by retaining one or two extra decimals in the coefficients.

For instance, suppose that an approximation to $\ln (1 + x)$ is required in the range $0 \leq x \leq 1$, accurate to three decimal places. Examination of the coefficients A_n in the series

$$\ln (1 + x) = \sum_{n=0}^{\infty} A_n T_n^*(x)$$

given in the tables, shows that $|A_n| < 0.0005$ when $n > 3$, and that

$$\sum_{n=4}^{\infty} |A_n| = 0.000503\cdots.$$

Hence

$$\begin{aligned}
\ln (1 + x) \doteq\ & 0.37645 + 0.34315(2x - 1) - 0.02944(8x^2 - 8x + 1) \\
& + 0.00337(32x^3 - 48x^2 + 18x - 1) \\
=\ & 0.00049 + 0.98248x - 0.39728x^2 + 0.10784x^3
\end{aligned}$$

with a maximum error not exceeding 0.00053.

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[1] *Tables of the Chebyshev Polynomials* $S_n(x)$ *and* $C_n(x)$. NBS Applied Math. series 9, 1952.

Tables of Chebyshev coefficients

$\sin \frac{1}{2}\pi x$

n	A_n
0	1.27627 8962
1	−0.28526 1569
2	0.00911 8016
3	−0.00013 6587
4	0.00000 1185
5	−0.00000 0007

$$\sin \tfrac{1}{2}\pi x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2), \qquad -1 \le x \le 1.$$

$\cos \frac{1}{2}\pi x$

n	A_n
0	0.47200 1216
1	−0.49940 3258
2	0.02799 2080
3	−0.00059 6695
4	0.00000 6704
5	−0.00000 0047

$$\cos \tfrac{1}{2}\pi x = \sum_{n=0}^{\infty} A_n T_n^*(x^2), \qquad -1 \le x \le 1.$$

$\arctan x$

n	A_n
0	0.88137 3587
1	−0.10589 2925
2	0.01113 5843
3	−0.00138 1195
4	0.00018 5743
5	−0.00002 6215
6	0.00000 3821
7	−0.00000 0570
8	0.00000 0086
9	−0.00000 0013
10	0.00000 0002

$$\arctan x = x \sum_{n=0}^{\infty} A_n T_n^*(x^2), \qquad -1 \le x \le 1.$$

for $|x| > 1$, use

$$\arctan x = \tfrac{1}{2}\pi - \arctan (1/x)$$

$\arcsin x$, $\arccos x$

n	A_n
0	1.05123 1959
1	0.05494 6487
2	0.00408 0631
3	0.00040 7890
4	0.00004 6985
5	0.00000 5881
6	0.00000 0777
7	0.00000 0107
8	0.00000 0015
9	0.00000 0002

$$\arcsin x = x \sum_{n=0}^{\infty} A_n T_n^*(2x^2), \qquad -\tfrac{1}{2}\sqrt{2} \le x \le \tfrac{1}{2}\sqrt{2}.$$

$$\arccos x = \tfrac{1}{2}\pi - x \sum_{n=0}^{\infty} A_n T_n^*(2x^2), \qquad 0 \le x \le \tfrac{1}{2}\sqrt{2}.$$

For $\frac{1}{2}\sqrt{2} \le x \le 1$, use
$\arcsin x = \arccos (1 - x^2)^{\frac{1}{2}}$,
$\arccos x = \arcsin (1 - x^2)^{\frac{1}{2}}$.

e^x

n	A_n
0	1.75338 7654
1	0.85039 1654
2	0.10520 8694
3	0.00872 2105
4	0.00054 3437

e^{-x}

n	A_n
0	0.64503 5270
1	−0.31284 1606
2	0.03870 4116
3	−0.00320 8683
4	0.00019 9919

5	0.00002 7115	5	−0.00000 9975
6	0.00000 1128	6	0.00000 0415
7	0.00000 0040	7	−0.00000 0015
8	0.00000 0001		

$$e^x = \sum_{n=0}^{\infty} A_n T_n^*(x), \quad 0 \le x \le 1. \qquad e^{-x} = \sum_{n=0}^{\infty} A_n T_n^*(x), \quad 0 \le x \le 1.$$

	$\log(1+x)$		$\Gamma(1+x)$
n	A_n	n	A_n
0	0.37645 2813	0	0.94178 5598
1	0.34314 5750	1	0.00441 5381
2	−0.02943 7252	2	0.05685 0437
3	0.00336 7089	3	−0.00421 9835
4	−0.00043 3276	4	0.00132 6808
5	0.00005 9471	5	−0.00018 9303
6	−0.00000 8503	6	0.00003 6069
7	0.00000 1250	7	−0.00000 6057
8	−0.00000 0188	8	0.00000 1056
9	0.00000 0029	9	−0.00000 0181
10	−0.00000 0004	10	0.00000 0031
11	0.00000 0001	11	−0.00000 0005
		12	0.00000 0001

$$\log(1+x) = \sum_{n=0}^{\infty} A_n T_n^*(x), \qquad \Gamma(1+x) = \sum_{0=n}^{\infty} A_n T_n^*(x),$$

$$0 \le x \le 1. \qquad 0 \le x \le 1.$$

	$J_0(x)$		$J_1(x)$
n	A_n	n	A_n
0	0.03154 0613	0	0.06942 43523
1	−0.21461 6183	1	−0.11557 79057
2	0.00433 6620	2	0.12167 94099
3	−0.26620 3654	3	−0.11488 40465
4	0.30612 5520	4	0.05779 05331
5	−0.13638 8770	5	−0.01692 38801
6	0.03434 7540	6	0.00323 50252
7	−0.00569 8082	7	−0.00043 70609
8	0.00067 7504	8	0.00004 40991
9	−0.00006 0947	9	−0.00000 34583
10	0.00000 4309	10	0.00000 02172
11	−0.00000 0246	11	−0.00000 00112
12	0.00000 0012	12	0.00000 00005

$$J_0(x) = \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{x^2}{100}\right), \qquad -10 \le x \le 10.$$

$$J_1(x) = x \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{x^2}{100}\right), \qquad -10 \le x \le 10.$$

n	$T_n^*(x)$
0	1
1	$2x - 1$
2	$8x^2 - 8x + 1$
3	$32x^3 - 48x^2 + 18x - 1$
4	$128x^4 - 256x^3 + 160x^2 - 32x + 1$
5	$512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$
6	$2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1$
7	$8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9408x^3 - 1568x^2 + 98x - 1$
8	$32768x^8 - 1\ 31072x^7 + 2\ 12992x^6 - 1\ 80224x^5 + 84480x^4 - 21504x^3 + 2688x^2 - 128x + 1$
9	$1\ 31072x^9 - 5\ 89824x^8 + 11\ 05920x^7 - 11\ 18208x^6 + 6\ 58944x^5 - 2\ 28096x^4 + 44352x^3 - 4320x^2 + 162x - 1$
10	$5\ 24288x^{10} - 26\ 21440x^9 + 55\ 70560x^8 - 65\ 53600x^7 + 46\ 59200x^6 - 20\ 50048x^5 + 5\ 49120x^4 - 84480x^3 + 6600x^2 - 200x + 1$
11	$20\ 97152x^{11} - 115\ 34336x^{10} + 273\ 94048x^9 - 367\ 65696x^8 + 306\ 38080x^7 - 164\ 00384x^6 + 56\ 37632x^5 - 12\ 08064x^4 + 1\ 51008x^3 - 9680x^2 + 242x - 1$
12	$83\ 88608x^{12} - 503\ 31648x^{11} + 1321\ 20576x^{10} - 1992\ 29440x^9 + 1905\ 13152x^8 - 1203\ 24096x^7 + 506\ 92096x^6 - 140\ 57472x^5 + 24\ 71040x^4 - 2\ 56256x^3 + 13728x^2 - 288x + 1.$

RECENT MATHEMATICAL TABLES

1202[A,P].—M. L. CLINNICK, (a) *Gear Ratios No.* 43. (b) *Gear Ratios No.* 59. Privately printed, 3211 School Street, Oakland 2, California, 1953. Each book has 84 unnumbered pages 8.9 × 11.4 cm. and 8.26 × 14.0 cm. respectively, photo-offset from typescript. Price $1.00 each.

These pocket-sized tables are designed for use in selecting appropriate sprocket gears in motorcycle racing events. They are triple entry tables giving 2D values of

$$R = kr/(ec)$$

for $c = 10(1)23$, $e = 15(1)25$, $r = 46(1)75$, $k = 43$, 59. Unrealistic values of $R \ge 15$ are omitted. In the intended application r, e and c are the numbers of teeth in the rear, engine, and countershaft sprockets respectively. The clutch sprocket is assumed to have 43 teeth or 59 corresponding to certain popular British and American makes of motorcycle. Instructions are given in (b).

D. H. L.

1203[A,B,P].—J. K. LYNCH, *Kilocycle-Radian Frequency Conversion Tables.* Commonwealth of Australia, Postmaster-general's Dept., Research Laboratories, *Report* No. 3726. Melbourne, 1953, 24 p. 20.2 × 25.4 cm.

This table gives 6S values of

$$\omega = 2000\pi F$$

and ω^3, ω^{-1} and ω^{-2} for

$$F = .1(.1)100(10)1000, \quad 10^k \text{ for } k = 6(1)11.$$

This allows the electrical engineer to pass easily from a frequency F in kilocycles per second to the corresponding angular velocity ω in radians per second, or to its square, reciprocal or reciprocal square.

For example the inductance L and capacitance C for resonance of frequency F are related by $L = \omega^{-2}/C$.

D. H. L.

1204[A,D].—G. E. REYNOLDS, *Conversion Tables of Tangents or Cotangents to Sines and Cosines of Three Decimals.* Air Force Cambridge Research Center, *Technical Report* 53–29, Cambridge, Mass., 1953. 26 p. 18.4 × 27.3 cm.

This is a short handy table of sines and cosines as functions of tangents or cotangents. Values of sines and cosines are given to 3D as consecutive multiples of 10^{-3}. Corresponding 4S values of tangents and cotangents are listed alongside. The table is intended to be entered via the more rapidly moving functions tan and cot to read out corresponding values of sine and cos without interpolation.

The table is in fact a table of $r/(1 - r^2)^{\frac{1}{2}}$, $r - .0005 = 0(.001)1$. Applications are mentioned to the layout of parallel plate surfaces and the path of a milling cutter.

D. H. L.

1205[B,F].—D. R. KAPREKAR, *Cycles of Recurring Decimals*, v. II. (*From* N = 167 *to* 213 *and many other numbers.*) Khare Wada, Deolali, India, 1953. Published by the author, iv + 47 p., 24.3 × 16.7 cm. Price 6 rupees.

This is an extension of v. I, reviewed in *MTAC*, v. 7, p. 238, to the extent mentioned in the title. Besides the table of cycles there are other tables as follows.

P. 25. Table of 2^n and 5^n for $n = 1(1)33$

P. 36–41. Table of exponents of 10 (mod p) for all primes $p \leq 13709$.

This table is taken from a previous table of KRAITCHIK.[1] This reprint contains three errata.

p. 36 *For* $P = 669$ *read* $P = 659$
p. 40 $P = 9619$, *for* $C = 8$ *read* $C = 3$
p. 41 *For* $P = 12901$ *read* $P = 12907$

P. 43–47. Table of factors of $(10^e - 1)/p$ where e is the exponent of 10 (mod p) for 23 primes p mostly less than 100. In connection with $p = 47$ the author claims the discovery of the prime

19342359767891682 7853.

However on the next page two factors of this number are given, namely 2531 and 549797184491917, the last being misprinted.

D. H. L.

[1] M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 1. Paris, 1924, p. 131–145.

1206[C,D].—Akademiîa Nauk SSSR. Institut Tochnoĭ mekhaniki i vychislitel'noĭ techniki. Matematicheskie Tablit͡sy. *Desi͡atiznachnye Tablit͡sy logarifmov kompleksnykh chisel i perekhoda ot dekartavykh koordinat k polit͡arnym. Tablit͡sy funkt͡siĭ.* [Ten place tables of logarithms of complex numbers and of the transformation from cartesian to polar coordinates. Tables of functions] $\ln x$, arctg x, $\frac{1}{2}\ln(1+x^2)$, $(1+x^2)^{\frac{1}{2}}$. Moscow, 1952. 116 p. 17 × 26 cm. 10.8 roubles.

There are four 10D tables each at interval .001 and with Δ^2:

A. $\ln x$, $1 \leq x < 10$;
B. $\frac{1}{2}\ln(1+x^2)$, $0 \leq x \leq 1$;
C. arc tan x, $0 \leq x \leq 1$;
D. $(1+x^2)^{\frac{1}{2}}$, $0 \leq x \leq 1$.

From such tables $\ln z = \ln(A+iB) = \ln A + \frac{1}{2}\ln[1+(B/A)^2] + i \arctan(B/A)$, $|B| < |A|$, $r = |A|[1+(B/A)^2]$, $\theta = \frac{1}{2}\pi - \arctan(B/A)$ may be found; similar expressions exist for $|B| > |A|$, since $\ln(A+iB) = \ln i + \ln(B-iA)$.

Illustrative numerical examples in the use of the tables are given on pages 4–6, and there is an errata slip with 15 corrections in the following tables. An interpolation sheet is in a pocket of the volume, of which there were 3000 copies in the edition.

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1207[D].—H. E. Salzer, "Radix table for obtaining hyperbolic and inverse hyperbolic functions to many places," *Jn. Math. Phys.*, v. 32, 1953, p. 197–202.

These tables are intended to give data from which 18D values of hyperbolic functions and their inverses can be found with an ordinary desk calculator. A similar table for circular functions has been given by the author in *MTAC*, v. 5, p. 9–11. As in the previous radix table, the author chooses the inverse tangent function. In this case it is

$$\text{Arc tanh}\,(k \cdot 10^{-\lambda}) \quad k = 1(1)9, \quad \lambda = 1(1)6.$$

These values are given to 20D. He also gives 20D values of

$$\tanh A \text{ for } A = 1(1)24$$

and

$$\tfrac{1}{2}\ln x \text{ for } x = 1.2(.2)2(1)10.$$

By using the addition theorems for tanh and arc tanh and approximations to these functions for small values of the argument any desired value can be built up. Of course, the other hyperbolic functions and their inverses can be expressed in terms of the tangent and its inverse.

D. H. L.

1208[F].—Ove Hemer, "Note on the diophantine equation $y^2 - k = x^3$," *Arkiv för Matematik*, v. 3, 1954, p. 67–77.

This paper contains corrections and additions to the author's dissertation [*MTAC*, v. 7, p. 86].

Tables 1 and 2 of this dissertation are reprinted here with the corrections called for in our previous review. Table 1 is now supposed to be complete in the sense that it gives all solutions of the equation $y^2 - k = x^3$ for $0 < k < 100$, whenever the equation is soluble, together with the number N of solutions. The largest $N = 8$ is for the well known case of $k = 17$ discussed by MORDELL, Table 2 for $-100 < k < 0$ still contains 22 incomplete cases. For these, the coefficients and solutions of the corresponding cubic forms are given. Table 3 of the dissertation is not reproduced here. Table 4 is slightly enlarged giving the discriminant $-D$, the fundamental rings and units in the cubic field corresponding to all square free values of $1 < k < 50$, and for the values of k corresponding to soluble equations for $50 < k < 100$. There still remain 7 cases in which the unit is "not definitely proved to be fundamental."

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1209[F].—D. H. LEHMER, EMMA LEHMER, & H. S. VANDIVER, "An application of high speed computing to Fermat's Last Theorem," Nat. Acad. of Sci., *Proc.*, v. 50, 1954, p. 25–33.

Kummer defined a prime l to be regular if it does not divide any of the first $(l - 3)/2$ Bernoulli numbers and showed that

$$x^l + y^l = z^l, \quad l > 2$$

is impossible in non-zero integers if l is a regular prime. In this paper, the irregular primes less than 2000 are tabulated, together with the accompanying least prime p of the form $1 + kl$, as well as three other associated constants; $2a$, Q_a, $Q_a{}^k$. On the basis of this table not only is Fermat's Theorem established for $l < 2000$, but data are given which will greatly simplify and facilitate the study of units and ideals in cyclotomic fields.

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1210[G,K].—F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions, Part IV," *Biometrika*, v. 40, 1953, p. 427–446.

In this set of tables the authors continue a series of fundamental symmetric function tables for functions of weight not exceeding twelve. When the series is complete the user will be able to relate any two of the following kinds of symmetric functions of x_i:

(U) unitary, or elementary, $a_k = \sum x_1 x_2 \cdots x_k$
(S) sums of like powers, $S_k = \sum x_1{}^k$
(M) monomial, such as $(3, 2, 1, 1) = \sum x_1{}^3 x_2{}^2 x_3 x_4$
(H) homogeneous product-sums h_r generated by

$$\prod_i (1 - x_i t)^{-1} = \sum_{r=0}^{\infty} h_r t^r.$$

The first three parts[1] have already been published and are described in RMT **769** and **1020** [*MTAC*, v. 4, p. 146, v. 6, p. 224–5, see also corri-

gendum, v. 8, p. 188]. These relate S and M, M and U, and U and H respectively. The present Part IV relates M and H. Thus one may use the table of weight 3 to read such results as

$$h_1h_2 = (3) + 2(2, 1) + 3(1, 1, 1)$$

and

$$(2, 1) = -2h_1^3 + 5h_1h_2 - 3h_3.$$

In the arrangement of the tables advantage is taken of Hirsch's law of symmetry which states that the coefficient of $(p, q, r, \cdots)$ in the expansion of $h_ah_bh_c\cdots$ is the same as the coefficient of $(a, b, c, \cdots)$ in the expansion of $h_ph_qh_r\cdots$. Thus although each expanded symmetric function of weight w requires w terms each coefficient may be used twice. The whole set of tables for Part IV occupies no more space than the previously published parts which are triangular in nature. The printing is in the same small but elegant type of the earlier tables.

Part V which will relate symmetric functions of types U and S will complete the set since a Part VI relating H and S would be identical with Part V except for obvious sign changes.

D. H. L.

[1] F. N. David & M. G. Kendall, "Tables of symmetric functions Part I; Parts II and III," *Biometrika*, v. 36, 1949, p. 431–449; v. 38, 1951, p. 435–462.

1211[K].—R. S. Burington & D. C. May, Jr., *Handbook of Probability and Statistics with Tables*. Handbook Publishers, Sandusky, Ohio, 1953. ix + 332 p., 14.3 × 20.3 cm. Price $4.50.

This book is principally a rather complete and up-to-date handbook of statistical methods and theory which will prove very useful; reviews concerned with this aspect will be found elsewhere. Scattered through the text are some 17 tables and at the end 24 tables and a nomogram occupy 62 pages.

The tables in the text, mostly short, begin with four giving ordinates, areas, and percentage points for the normal frequency function. The following four are devoted to the normal bivariate frequency function. The first, if this function be written $f(x, y) = k \exp(-c^2/2)$, gives c to 4S for the $P\%$ probability ellipse for $P = 25, 50, 75, 90, 95, 99$. The next three deal with the circular case, giving radii $c\sigma$ of $P\%$ probability circles with c to 4S for $P = 25, 39.3, 50, 54.4, 75, 90, 95, 99$; the radii R_1 to 4D in standard units of a circular disk centered at the means over which the probability integral is P for $P = 0(.05)1$ as well as the integrand evaluated at R_1 to 4D; and values of the same probability integral to 3 or 4D over a circular disk of radius R/σ centered at the distance d/σ from the means for $R/\sigma = .1(.1)1(.2)3$ and $d/\sigma = 0(.1)3(.2)6$. The next two tables are concerned with the trivariate normal frequency function with zero correlations, the first being the exact analog of the first bivariate table and the second, for the spherical case, the radius $c\sigma$ (c to 3D) of the sphere within which the total frequency is p for $p = .25, .5, .75, .9, .95, .99$.

All but one of the remaining tables in the text are related to sampling distributions. Those for which credit is given to other sources are here designated by (R). If s is the standard deviation in a sample of n from a normal universe with standard deviation σ, the next table (R) gives b for which $P(s > b\sigma) = P$ to 3D for $n = 5(1)30$ and $P = .01, .05, .95, .99$.

If $r = r_z\sqrt{n-1}$ where r_{xy} is the coefficient of correlation in a sample of n) from an$_y$ uncorrelated bivariate normal universe, the following table (R gives values of r_1 to 3D for which $P(r \geqq r_1) = P$ for $P = .001, .01, .05, .1$ and $m = n - 2 = 1(1)30(5)50(10)100, 120, \infty$. Then comes a table ($R$) concerned with the distribution of ranges, R, in samples of n from the same universe. Values of $E\left(\frac{R}{\sigma}\right)$ are given to 4S for $n = 2(1)20$, to 3S for $n = 30, 50, 75, 100$, and to 2S for $n = 150, 200$ and $\sigma_{R/\sigma}$ is given to 3S for $n = 2(1)20$. Also R'/σ is given to 2D for which $P(R \leqq R') = P$ for $P = .001, .005, .01, .025, .05, .1, .9, .95, .975, .995, .999$ and $n = 2(1)20$. Next the probability p that the fraction h of any continuous universe is included in the range of a sample of n drawn from it is tabulated to 2S (more in a few cases) for $h = .8, .9, .95, .99$ and $n = 5(5)30, 40, 50, 75, 100$. The next pair of tables (R) give equal-tail confidence intervals for the probability of success p in a binomial universe from which n trials have yielded S successes. The end-points are given to 2D for $n = 10, 15, 20, 30, 50, 100$, $S = 0(1)50$ and the confidence coefficients .95 and .99. Then follows a table of Stirling's numbers of the first kind, S_i^n, for $n = n(1)10$ and $i = 0(1)9$. The final table (R) in the text gives the numerical coefficients for estimating center lines and control limits for $\bar{X}$ and σ or R charts in quality control, both for process means and standard deviations known or unknown. These are given to 3 or 4S for samples of $n = 2(1)10(2)20$ for averages and standard deviations and $n = 2(1)10(2)14$ for ranges.

The section devoted to tables (p. 247–309) include:

Table I. $\binom{n}{x} p^x(1-p)^{n-x}$ to 4D for $p = .05(.05).5$ and $n = 1(1)20$.

Table II. $\sum_{x=x'}^{n} \binom{n}{x} p^x(1-p)^{n-x}$ to 4D for the same values of p and n as in Table I.

Table III. p for which the incomplete β-function, $I_p(x, n-x+1) = .005, .01, .025, .05, .1$ for $x = 1(1)15, 20, 30, 60, \infty$ and $n - x + 1 = 1(1)6, 10, 15, 20, 30, 60$.

Table IV. $\binom{n}{r}$ for $n = s(1)20$ and $r = 0(1)10$.

Table V. $[np(1-p)]^{\frac{1}{2}}$ to 4D for $n = 1(1)20$ and $p = .05(.05).5$.

Table VI. $(pq)^{\frac{1}{2}}$ to 4D for $p = .005(.005).5$.

Table VII. $e^{-m}m^x/x!$ for $x = 0(1)\cdots$, to 4D for $m = .1(.1)10(1)20$.

Table VIII. $\sum_{x=x'}^{\infty} e^{-m}m^x/x!$ for the same ranges of the arguments as in Table VII.

Table IX. $\psi(t) = (2\pi)^{-\frac{1}{2}}e^{-t^2/2}$, $\alpha/2 = \int_0^t \psi(\tau)d\tau$, and $\psi^{(n)}(t)$ to 4D or 4S for $t = 0(.01)4(.05)5$ and $n = 1(1)6$.

p. 275. Selected important constants to 8D and their common logarithms to 7 or 8D.

Table X(R). 1% and 5% points of the F distribution to 3 or more S for

the degrees of freedom 1(1)12, 14, 16, 20, 24, 30, 40, 50, 75, 100, 200, 500, ∞ for the larger estimated variance and 1(1)30(2)50(5)80, 100, 125, 150, 200, 400, 1000, ∞ for the smaller estimated variance.

Table XI(R). 1%, 1% and 5% points of the z-distribution. ($Z = \frac{1}{2} \ln F$) to 4D for the degrees of freedom 1(1)6, 8, 12, 24, ∞ for the larger estimated variance and 1(1)30, 60, ∞ for the smaller estimated variance.

Table XII(R). The percentage points 100ϵ of the Student-Fisher t distribution to 3D for $\epsilon = .001, .01, .02, .05, .1(.1).9$ and the degrees of freedom 1(1)30, 40, 60, 120, ∞.

Table XIII. The cumulative Student-Fisher t distribution to 3D for $t = 0(.2)5$ and the degrees of freedom 1(1)6, 8, 10, 15, 20.

p. 285. A nomogram for $1 - (1 - p)^n$.

Table XIV. (R)χ_0^2 satisfying $P(\chi^2 \geqq \chi_0^2) = \epsilon$ to 3D for $\epsilon = .001, .01, .02, .05, .1, .2, .3, .5, .7, .8, .9, .95, .99$ and the degrees of freedom 1(1)30.

Table XV. e^{-x} to 5D for $x = 0(.01)3(.05)4(.1)6(.25)7(.5)10$.

Table XVI. $n!$ to 5S and $\log_{10}(n!)$ to 5D for $n = 1(1)100$.

Table XVII. $\Gamma(n)$ to 4D for $n = 1.01(.01)4.99$.

Table XVIII. $\log_{10}\Gamma(n)$ to 4D for $n = 1.01(.01)2$.

Table XIX. $n!$ exact and $(n!)^{-1}$ to 5S for $n = 1(1)20$.

Table XX. n^2 exact and $n^{\frac{1}{2}}$, $(10n)^{\frac{1}{2}}$ and $1000\,n^{-1}$ to 5S for $n = 1(1)999$.

Table XXI. The natural trigonometric functions and the radian measure to 4D for the angle $\alpha = 0(1°)90°$.

Table XXII. $\ln N$ to 3D for $N = 0(.01).99$ and to 5D for $N = 1(.01)10.09$.

Table XXIII. $\log_{10} N$ to 4D for $N = 100(1)999$.

C. C. C.

1212[K].—J. H. CADWELL, "The distribution of quasi-ranges in samples from a normal population," *Annals Math. Stat.*, v. 24, 1953, p. 603–613.

Let $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ be the ordered values in a random sample of size n drawn from a normal population. The statistic $w_r = x_{n-r} - x_{r+1}$ is called the range for $r = 0$, and a quasi-range for $r \geqq 1$. The author gives an approximate, but apparently very accurate, method for evaluating the probability distribution function of w_r. Of interest are the following results: (1) in section 3, the approximation for the p.d.f. of w_r is given; (2) in section 4, a comparison is made between the exact and approximate values of $E(w_0)$ and $E(w_1)$ for $n = 2, 8, 20, 30, 60, 100$; (3) in section 5, the mean, variance, skewness, and flatness of the range are given for $n = 20$; (4) in section 7, there is a brief discussion of the efficiency of w_0, w_1, and w_2 as estimators of the population standard deviation σ; (5) in section 8, a table of values of moment constants to 3 or 4D and the .1, 1, 2.5, 5, 95, 97.5, 99, and 99.9 percentage points of w_1 are given for $n = 10(1)30$ to 2D; (6) in section 9, there is a brief consideration of the efficiency of estimates of σ based on a linear combination of two quasi-ranges. The paper fills in a gap between published work for cases where $n \leqq 10$ on the one hand and n large on the other.

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1213[K].—R. E. CLARK, "Percentage points of the incomplete beta function," Am. Stat. Assn., *Jn.*, v. 48, 1953, p. 831–843.

The table presented here gives to 4S the value of

$$p = P(N, X, \alpha)$$

defined by

$$\alpha = \sum_{r=X}^{N} \binom{N}{r} p^r (1-p)^{N-r} = I_p(X, N - X + 1)$$

for

$$N = 10(1)50, \quad X = 1(1)N, \quad \alpha = .005, .01, .025, .05,$$

where $I_p(X, Y)$ is KARL PEARSON's incomplete beta function. Applications of the table are indicated.

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1214[K].—W. J. DIXON, "Processing data for outliers," *Biometrics*, v. 9, 1953, p. 74–89.

Samples of size N are drawn from a mixture of normal distributions $(1-\gamma)N(\mu, \sigma^2) + \gamma N(\mu + \lambda\sigma, \sigma^2)$ and $(1-\gamma)N(\mu, \sigma^2) + \gamma N(\mu, \lambda^2\sigma^2)$, respectively, in which γ measures the contamination of the $N(\mu, \sigma^2)$ universe. Comparisons are made between the mean and median as estimates of μ; range and variance as estimates of dispersion. The bias and the MSE (mean squared error) for untreated (treatment refers to processing the data in a sample to remove outliers) samples (Table I), the ratios of the MSE of the mean and median of untreated data to the MSE of the mean for uncontaminated data (Table II), and minimum MSE of four procedures: use of $\bar{X}$ after treating for rejection at the significance levels, $\alpha = .01, .05, .10$ and the use of median (Table III), serve as a basis for comparing the mean and median. The appropriate α to remove bias in s^2 and range (Tables IV, V, VI, VII) are used to compare range and variance. For most of the Tables $N = 5, 15$; $\gamma = .01, .05, .10, .20$, $\lambda = 0, 2, 3, 5, 7$ with some modifications. The values known to be correct are those for $\lambda = 0$, the quantities for the mean in Table I, the results for no contamination in Tables II and III and the first line in Table V. Other results for $N = 5, 15$ are based on 100 and 66 samples, respectively. The appendix gives critical values and criteria for testing for extreme values based on $r_{ij} = (X_N - X_{N-i})/(X_N - X_{j+1})$, $\alpha = .005, .01, .05, .10, .20, .30$, $N = 3(1)7$ for r_{10}, $N = 8, 9, 10$ for r_{11}, $N = 11, 12, 13$ for r_{21}, $N = 14(1)25$ for r_{22} to 3D.

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1215[K].—ABRAHAM GOLUB, "Designing single-sampling inspection plans when the size is fixed," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 278–288.

Consider the sample size n fixed for a given single sampling plan. The problem is to choose an acceptance number s which yields the "best" protection against misclassifying submitted lots. Two cases are considered: (1) single sampling-plans for placing a lot into one of two categories defined

by fraction defectives p_1 and p_2, (2) single sampling-plans for placing a lot into one of m categories defined by fraction defectives $p_1, p_2, \cdots, p_m$. In case (1), the "best" plans are defined as those which minimize the sum of the producer's and consumer's risks. In case (2), the "best" plans are defined as those which minimize the sum of the probabilities of misclassifying the submitted lots. Given m categories defined by fraction defectives $p_1, p_2, \cdots, p_m$, Tables 1 through 8 provide $m - 1$ acceptance numbers, c_i $(i = 1, 2, \cdots, m - 1)$ for single sampling-plans based on given fixed sample sizes 5(5)40, for acceptable lot qualities .01(.01).20, and for objectionable lot qualities .01(.01).40 respectively. The above tables should be very useful in quality control applications in which it is desired to divide the lot quality into more than two categories.

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1216[K].—E. L. Grab & I. R. Savage, "Tables of the expected value of $1/x$ for positive Bernoulli and Poisson variables," Am. Stat. Assn., *Jn.*, v. 49, 1954, p. 169–177.

Let X be a random variable having the probability distribution

$$\text{(a)} \quad P\{X = k\} = \binom{n}{k} p^k q^{n-k}(1 - q^n)^{-1},$$

where $k = 1, 2, \cdots, n$; $q = 1 - p$; $0 < p \leqq 1$;

$$\text{(b)} \quad P\{X = k\} = e^{-m}(1 - e^{-m})^{-1} m^k / k!,$$

where $k = 1, 2, \cdots$; $m > 0$. For case (a) Table I gives the values of

$$E(1/X \mid n, p) = (1 - q^n)^{-1} \sum_{k=1}^{n} \binom{n}{k} k^{-1} p^k q^{n-k}$$

to 5D for $n = 2(1)20$, $p = .01$, $.05(.05).95$, $.99$; $n = 21(1)30$, $p = .01$, $.05(.05).50$. The authors give an empirical approximation to $(np - q)^{-1}$ for certain values of the parameters which is accurate to at least 2*S*. For case (b) Table II gives the values of

$$E(1/X \mid m) = e^{-m}(1 - e^{-m})^{-1} \sum_{k=1}^{\infty} m^k k^{-1} / k!$$

to 5D for $m = .01$, $.05$, $(.05)1(.1)2(.2)5(.5)7(1)10(2)20$.

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1217[K].—F. E. Grubbs & H. J. Coon, "On setting test limits relative to specification limits," *Industrial Quality Control*, v. 10, No. 5, 1954, p. 15–20.

This paper is concerned with the problem of setting test limits when specification limits are fixed and test measurements are subject to error. The authors consider test limits established in accordance with the following criteria: (1) producer's and consumer's risk are made equal; (2) the sum of

producer's and consumer's risk is minimized; and (3) the cost of incorrect decisions is minimized. As used here, consumer's risk and producer's risk do not have quite the same meaning that is usually associated with these terms in connection with sampling inspection. For the purposes of this paper, the consumer's risk, designated as A, is defined as the joint probability of selecting a defective unit and judging it acceptable. Similarly the producer's risk, designated as B, is defined as the joint probability of selecting an acceptable unit and judging it defective. With specification limits expressed as $\mu \pm k\sigma_x$ where σ_x is the product standard deviation, test limits are expressed as $\mu \pm (k\sigma_x - b\sigma_e)$ where σ_e is the standard deviation of the errors of measurement. For criteria (1) and (2) above, tables of b, A, B, and $A + B$ are given to 4D with $k = 1.5(.5)4$, and $r = .5(.25)1(1)10$, where $r = \sigma_x/\sigma_e$. Examples are given which illustrate the use of these tables.

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1218[K].—J. M. HAMMERSLEY, "On estimating restricted parameters," Roy. Stat. Soc., *Jn.*, s.B v. 12, 1950, p. 192–240.

This paper contains a table of solutions of the following distribution problem: A philanthropist with u coins gives the first beggar he meets a number of coins which with equal probabilities may be $1, 2, \cdots$, or u. If the first beggar receives $u - r$ coins, the next beggar will receive $1, 2, \cdots$, or r coins, each number being equally probable. If this is continued until the coins are exhausted, what is the probability that exactly ν beggars will receive a gift? The probability $P(u, \nu)$ satisfies the equation

$$P(u, \nu) = \sum_{r=\nu-1}^{u-1} u^{-1} P(r, \nu - 1)$$

with $P(u, 1) = 1/u$. $P(u, \nu)$ is tabled for u and $\nu = 1(1)13$.

C. C. C.

1219[K].—HANNES HYRENIUS, "On the use of ranges, cross-ranges, and extremes in comparing small samples," Am. Stat. Assn., *Jn.*, v. 48, 1953, p. 534–545.

A procedure is proposed using ranges, cross-ranges, and lower-extreme-differences as alternatives to the variance in testing variation homogeneity. The universe sampled is rectangular. Let sample 1 have N_1 items with lower extreme u_1 and upper extreme v_1, and let sample 2 have the corresponding N_2, u_2, and v_2. Designate the samples so that $u_1 \leqq u_2$. The ranges are $R_{11} = v_1 - u_1$ and $R_{22} = v_2 - u_2$. The cross-range is $R_{21} = v_2 - u_1$; and the lower-extreme-difference is $S_{21} = u_2 - u_1$.

Distributions of the test quotients $T = S_{21}(R_{11})^{-1}$, $U = R_{22}(R_{11})^{-1}$ and $V = R_{21}(R_{11})^{-1}$ are studied. With means and variances derived, tables are prepared showing upper 1, 5 and 10 percentage points of T, and showing upper .5, 2.5, and 5 and lower 95, 97.5 and 99.5 percentage points of U and V for N_1, $N_2 = 1(1)10$ to 2D.

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1220[K].—R. LATSCHA, "Tests of significance in a 2 × 2 contingency table: Extension of Finney's table," *Biometrika*, v. 40, 1953, p. 74–86.

Consider a 2 × 2 contingency table arranged so that the sum of the two values in the first row is A and the sum of the two values in the second row is B where $A = B$. Let a be value in the first row and first column while b is the value in the second row and first column. For given A, B, a, b, the table presented is entered in the section for A, the sub-section for B and the line for a; then the main body of the table shows in bold type the appropriate significance points for b. If the observed value b is equal to or less than the bold-faced integer in the column headed .05, .025, .01 or .005 (only values considered), then a/A is significantly greater than b/B (single-tail test) at these probability levels. This same condition on b furnishes a two-tail test of whether a/A differs from b/B at twice the probability value heading the column. A dash or absence of entry for specified A, B, a indicates that no 2 × 2 table in that class can show a significant effect at that level. The true significance level is at most equal to the probability value heading the column and in some cases may be noticeably smaller. For each table entry considered, the true probability value is given to 3D in small type next to the bold-faced integer with which b is compared. The table presented in this article covers the cases $A = 16(1)20$, $B = 2(1)A$, $A \geqq a \geqq$ (smallest integer not yielding all dashes). Previously FINNEY[1] obtained a similar table for $A = 9(1)15$.

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[1] D. J. FINNEY, "The Fisher-Yates test of significance in 2 × 2 contingency tables," *Biometrika*, v. 35, 1948, p. 145–156 [*MTAC*, v. 3, p. 359].

1221[K].—M. MASUYAMA & Y. KUROIWA, "Table for the likelihood solutions of gamma distribution and its medical applications," Union of Japanese Scientists and Engineers, *Reports of Statistical Application Research*, v. 1, 1951, p. 18–23.

For the Γ-type frequency function,

$$f(y) = \begin{cases} (y/a)^{p-1} \exp(-y/a)\, dy[a\Gamma(p)]^{-1} & \text{if } y \geqq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

the maximum likelihood estimates, $\hat{p}$ and $\hat{a}$, of the two parameters, p and a, are the solutions of the equations,

$$\ln \hat{p} - \psi(\hat{p}) = \ln (A/G) = g(\hat{p})$$
$$\hat{a} = A/\hat{p}$$

in which ψ is the digamma function and A and G are the arithmetic and geometric means of the sample respectively. To facilitate the approximate solution of the equations, the authors have tabulated $x \log (A/G)$, $g(x)$, $\log (A/G)$ and A/G to 8S or 7D for $x = .1(.05)3(.1)5(.5)10, 15, 20(10)50$. For large samples of N, $\frac{N\sigma_{\hat{a}}^2}{a^2}$ and $N\sigma_{\hat{p}}^2$ are given to 8D for the same values of x.

C. C. C.

1222[K].—NBSCL *Probability Tables for the Analysis of Extreme-Value Data*. NBS Applied Math. Ser. No. 22. U. S. Gov. Printing Office, Washington, D. C., 1953. iii + 32 p., 22 × 20 cm. Price $0.25.

There are many engineering and scientific problems in which the random variable being observed is a largest (smallest) or more generally m'th largest (m'th smallest) value. The 15 page introduction by E. J. GUMBEL, who pioneered in the field of extreme values, is a succinct and excellent survey of current knowledge in the field. In the first 9 pages of this introduction he sketches the theory and carries out the analysis of a problem in detail. The remainder of the introduction is devoted to listing the functions tabulated, giving a history of the genesis of the tables, and describing methods of computation. A useful bibliography completes the introduction.

If one carries out a suitable transformation on the largest value in a sample one gets a random variable which is called the reduced largest value. The asymptotic distribution of this reduced largest value is (for a wide class of common distributions such as the exponential, normal, and chi-square distributions) given by the cumulative distribution function $\Phi_y \equiv \exp(-e^{-y})$ and associated probability density function $\varphi_y \equiv \Phi_y' \equiv \exp(-y - e^{-y})$. Table 1 gives Φ_y and φ_y for $y = -3(.1) - 2.4(.05)0(.1)4(.2)8(.5)17$ to 7D. Table 2 gives the inverse of the cumulative probability function of extremes $y = -\log_e(-\log_e \Phi_y)$ for $\Phi_y = .0001(.0001).005(.001).988(.0001).9994(.00001).99999$ to 5D. Table 3 gives the probability density function φ_y as a function of the cumulative distribution function $\Phi_y[\varphi_y = -\Phi_y \log \Phi_y]$ for $\Phi_y = 0(.0001).01(.001).999$ to 5D. The reduced m'th largest value has associated cumulative distribution function $\Phi_m \equiv \int_{-\infty}^{y_m} \frac{m^m}{(m-1)!} \times \exp(-my - me^{-y})dy$. Table 4 gives probability points y_m for the reduced m'th largest value, $y_m = y_m(\Phi_m)$, for $m = 1(1)15(5)50$; $\Phi = .005, .01, .025, .05, .1, .25, .5, .75, .9, .95, .975, .99, .995$ to 5D. Tables 5 and 6 deal with an appropriately reduced range R. The cumulative probability distribution function $\Psi(R)$ is given by $\Psi(R) \equiv 2e^{-\frac{1}{2}R}K_1(2e^{-\frac{1}{2}R})$ and the probability density function $\psi(R) \equiv \Psi'(R)$ is given by $\psi(R) \equiv 2e^{-R}K_0(2e^{-\frac{1}{2}R})$. K_0 and K_1 are the modified Bessel functions of the second kind of orders 0 and 1, respectively. Table 5 gives $\Psi(R)$ and $\psi(R)$ for $R = -4.6(.1) - 3.3(.05)11(.5)20$ to 7D. In Table 6, the reduced range R is considered as a function of Ψ, i.e., as the solution of the equation $\Psi = 2e^{-\frac{1}{2}R}K_1(2e^{\frac{1}{2}R})$. This is given in Table 6 for $\Psi = .0001(.0001).001(.001).01(.01).95(.001).999(.0001).9999$ to 4D, except for $\Psi = .0001$ and $\geq .999$ to 3D.

The tables are the fruits of the painstaking labors of many people. Chief among these are E. J. GUMBEL who took part in the detailed planning, J. ARTHUR GREENWOOD, JULIUS LIEBLEIN, and H. E. SALZER. The preparation of the tables in their final form was carried out under SALZER's direction at the Computation Laboratory of the National Bureau of Standards. LIEBLEIN assisted in the preparation of the Introduction.

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1223[K].—NBSCL *Tables to Facilitate Sequential t-Tests*. NBS Appl. Math. Ser. No. 7. U. S. Gov. Printing Office, Washington, 1951. xix + 52 p. Price $0.45.

The sequential t-test is designed to test the hypothesis that the mean of a normally distributed variate with unknown mean and unknown variance has a given value. The present tables are designed for use with a probability ratio test based on the likelihood ratio for the non-central t to the central t, which thus differs from the one originally proposed by WALD.[1] However by a simple adjustment of the parameters these tables can also be used for Wald's test.

The test criterion after the n-th observation is $Z_n = [\sum_{\alpha=1}^{n}(x_\alpha - l_0)]^2 \times [\sum_{\alpha=1}^{n}(x_\alpha - l_0)^2]^{-1}$ in which l_0 is the value of the mean specified by the null hypothesis. The critical values of Z_n are solutions of the equation $L = \ln F\left(n/2, \tfrac{1}{2}; \frac{\delta^2 Z}{2}\right) - \frac{n\delta^2}{2}$ in which $F(a, b; x)$ is the confluent hypergeometric function,

$$\sum_{j=0}^{\infty} \frac{\Gamma(b)\Gamma(a+j)}{\Gamma(a)\Gamma(b+j)} \frac{x^j}{j!};$$

δ is the amount in standard units by which the mean under the alternate hypothesis differs from l_0 and L is either $\ln\{(1-\beta)/\alpha\}$ or $\ln\{\beta/(1-\alpha)\}$ in which α and β are the risks of errors of the first and second kinds respectively. Values of Z_n are given to 1D for $\delta = .1$, to 2D for $\delta = .2(.1).5$ (to 3D for $L < 0$ and $\delta = .2$) and to 3D for $\delta = .6(.1)1(.2)2$, 2.5 for $\pm L = 2$, ln 19, 3, 4, ln 99, 5, 6, 7 and $n = 1(1) \leqq 200$. (Values of n are carried as far in each case as is judged likely to be useful in practice but never beyond 200.) There are some smaller auxiliary tables. One of these gives the values of α and β to 3S for the L's listed; others are to assist in reaching a decision if n goes beyond the values in the table; and still others give approximate upper bounds for the error in linear interpolation with respect to L or S.

C. C. C.

[1] A. WALD, *Sequential Analysis*. New York, 1947, p. 204–207.

1224[K].—B. M. SEELBINDER, "On Stein's two-stage sampling scheme," *Annals Math. Stat.*, v. 24, 1953, p. 640–649.

STEIN[1] has proposed a two-stage plan for sampling from a normal population of unknown variance σ^2 in order to estimate the expectation with a confidence interval of preassigned length $2d$ and confidence coefficient $1 - \alpha$. The experimenter sets the size n_1 of the first sample, while the size $n - n_1$ of the second sample depends on the variability observed in the first. The present paper is concerned with the problem of choosing n_1 with an eye to minimizing $E(n)$. SEELBINDER tables $E(n)$ for some of the following values: $\alpha = .1, .05, .02, .01$; $d/\sigma = .01(.01).1(.1)1$; $n_1 - 1 = 5, 10(10)60, 80, 120,$

240. An approximation based on the normal distribution is found to be adequate.

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[1] CHARLES STEIN, "A two-sample test for a linear hypothesis whose power is independent of the variance," *Annals Math. Stat.*, v. 16, 1945, p. 243–258.

1225[K,N].—H. STEINHAUS, *Table of shuffled four-digit numbers. Rozprawy Matematyczne*, No. 6, Warsaw 1954, 46 p. The introduction is in Polish, Russian and English.

This table of 10,000 random 4-digit numbers differs from the usual random number lists in that no two 4-digit numbers are equal. In other words the table, when read in the usual order, gives one of the 10,000! permutations of the numbers 0000–9999. This feature allows the list to be used in problems in which samples are drawn without replacement. One may also read the digits vertically by fours and obtain samples which are drawn with replacement.

The table was produced by hand. To begin with a table of 10,000 four-digit numbers $U_n (n = 0(1)9999)$ was produced by

$$U_n \equiv 4567 U_{n-1} \pmod{10{,}000}$$

with $U_0 = 0000$. This table was arranged in a square 100×100 and then subjected to a number of randomizing transformations fully described in the introduction. Whether the 25 pages which finally resulted pass any of the standard tests of randomness is not indicated. Apparently the author has made no tests whatever. The reviewer subjected the first page to a frequency test which gave a χ^2 with a probability of only 2 percent.

The list could be useful for ordering the retirement of bonds.

D. H. L.

1226[K].—G. M. THOMPSON, "Scale factors and degrees of freedom for small sample sizes for χ approximation to the range," *Biometrika*, v. 40, 1953, p. 449–450.

Consider m independent samples of size n from a normal population, mean μ, variance σ^2. Let ω be the range of a sample, $\bar{\omega}_{m,n}$ the mean range. PATNAIK[1] found the approximation

$$\bar{\omega}_{m,n} = c\chi\nu^{-\frac{1}{2}}\sigma$$

where c is a scale factor, ν the degrees of freedom for χ. Evidently $\bar{\omega}_{m,n/c}$ is equivalent to the usual standard deviation estimator with ν degrees of freedom. Thompson tables c and ν to 4D for $m = 1$, $n = 2(1)10$. He also gives the 95% confidence limits on μ using ω, i.e. $\bar{X} \pm k_{.05}\omega$ to 4D, for $n = 2(1)10$. Agreement with results obtained by the use of Student's t is excellent.

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[1] P. B. PATNAIK, "The use of mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

1227[L].—F. DiMaggio, A. Gomza, W. E. Thomas & M. G. Salvadori, "Instabilità laterale di travi inflesse e compresse," Accad. Naz. dei Lincei, *Atti Rend.*, s. 8, v. 12, 1952, p. 524–529.

This paper summarizes the results of a number of technical reports published by the Dept. of Civil Engineering, Columbia University. The critical loads for the lateral buckling of several types of beams and loadings are given. These are the lowest eigenvalues of $\delta V = 0$, where

$$V = \tfrac{1}{2}B\int_0^l (\mu'')^2 dz + \tfrac{1}{4}Dh^2\int_0^l (\beta'')^2 dz + \tfrac{1}{2}H\int_0^l (\beta')^2 dz + \int_0^l M\beta\mu'' dz - \tfrac{1}{2}P\int_0^l (\mu')^2 dz.$$

B, D, h, H and P are constants. Table 1 gives the results for I-beams under unequal end moments, tabulates eigenvalues $K = M_2 l/(BC)^{\frac{1}{2}}$ to 2D as a function of $l^2/a^2 = 2l^2C/h^2D = .1$ (irreg.) ∞ and $r = M_1/M_2 = -1(.1)1$, where $H = C - PI_p/A$, $M(z) = M_2[r + (1-r)z/l]$, $M_1 = rM_2$, $\mu(0) = \mu''(0) = \mu(l) = \mu''(l) = 0$, $\beta(0) = \beta''(0) = \beta(l) = \beta''(l) = 0$. Table 2 gives the solution of rectangular beams under bending and compression, corresponding to $D = 0$, $\mu(0) = \mu(l) = \mu''(0) = \mu''(l) = 0$, $\beta(0) = \beta(l) = 0$. $K = M_2 l/(BH)^{\frac{1}{2}}$ given to 2D for $p = Pl^2\pi^{-2}/B = 0(.2)1$; and $r = M_1/M_2 = -1(.1)1$. Table 3 gives solution to rectangular cantilever beam with shear and axial load at the free end corresponding to $D = 0$; $r = 0$; $M(z) = Qz = M_2 z/l$ and $\mu(l) = \mu'(l) = \mu''(0) = B\mu''(0) + P\mu'(0) + Q\beta(0) = 0$, $\beta(l) = \beta'(0) = 0$, $q = Ql^2/(4.013(BH)^{\frac{1}{2}})$ given to 4D for $p = 4Pl^2\pi^{-2}/B = 1(-.2) - .4, -1$. Solutions of this problem (originated by Prandtl and Michell) for many other end conditions have been discussed in Timoshenko.[1]

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[1] S. Timoshenko, *Theory of Elastic Stability*. New York, 1936.

1228[L].—Stig Ekelöf, "Theory of electromagnetically delayed telephone relays," *Ericson Technics*, v. 9, 1953, p. 141–224.

On p. 221–224 there is a 4D table of the functions

$$\cosh x,\quad \frac{\sinh x}{x},\quad \frac{3\cosh x}{x^2} - \frac{3\sinh x}{x^3},\quad \frac{2(\cosh x - 1)}{x^2},$$
$$\frac{12\sinh x}{x^3} - \frac{24(\cosh x - 1)}{x^4}$$

for $x = 0(.01)1\cdot 5$. Graphs of these functions (in the same range) are given on p. 161.

A. E.

1229[L].—ERNST GLOWATZKI, "Tafel der Jacobischen elliptischen Funktion $\phi = \text{am}\,(mK/n)$," Bayer. Akad. Wiss., *math.-naturw. Kl., Abhandlungen*, Neue Folge, heft 61, 1953, 27 p.

3D tables of $\text{am}(u, k)$ in degrees for $u = mK/n$, $k = \sin\theta$, $n = 2(1)12$, $n = 1(1)n - 1$, $\theta^\circ = 0(1)90$. First differences are also given. The entries were computed from Legendre's tables, and were checked against the *Smithsonian Elliptic Function Tables* (see *MTAC*, v. 3, 1948–49, p. 89, RMT 485).

A. E.

1230[L].—LOUIS ROBIN & ALFREDO PEREIRA-GOMES, "L'antenne biconique, symmétrique, d'angle quelconque," *Ann. d. Télécommunications*, v. 8, 1953, p. 382–390.

The authors discuss the equations

$$P_n(\cos\theta) + P_n(-\cos\theta) = 0 \quad \text{or} \quad F\left(-\frac{n}{2}, \frac{n+1}{2}; \tfrac{1}{2}; \cos^2\theta\right) = 0$$

$$P_n(\cos\theta) - P_n(-\cos\theta) = 0 \quad \text{or} \quad F\left(\frac{1-n}{2}, \frac{n+2}{2}; \tfrac{3}{2}; \cos^2\theta\right) = 0$$

where P_n is the Legendre function, F the hypergeometric series, θ is given, and n is to be determined.

On p. 390 they give tables (mostly to 3D) of the first nine roots of the first equation, and of the first eight roots of the second equation, when $\theta = \pi/12, \pi/6, \pi/4, \pi/3, 5\pi/12$.

A. E.

1231[L].—M. ROTHMAN, "The problem of an infinite plate under an inclined loading, with tables of the integrals of $\text{Ai}(\pm x)$ and $\text{Bi}(\pm x)$," *Quart. Jn. Mech. Appl. Math.*, v. 7, 1954, p. 1–7.

For the definition of the Airy integrals $\text{Ai}(x)$ and $\text{Bi}(x)$ see *MTAC*, v. 1, 1944, p. 236.

Table 1. 7D values of $\int_0^x \text{Ai}(t)dt$ for $x = 0(.1)7.5$ with δ_m^2. 7D values of $\int_0^x \text{Bi}(t)dt$ for $x = 0(.1)2$ with δ_m^2 and γ^4.

Table 2. 7D values of $\int_0^x \text{Ai}(-t)dt$ for $x = 0(.1)10$ with δ_m^2 and γ^4.

Table 3. 7D values of $\int_0^x \text{Bi}(-t)dt$ for $x = 0(.1)10$ with δ_m^2 and γ^4.

A. E.

1232[L].—F. G. TRICOMI, "On the statistical distribution of mutant bacteria," *Bull. Mathem. Biophysics*, v. 15, 1953, p. 277–292.

The author gives 5D tables of

$$G(t) = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}\frac{t^n}{n!}, \quad G'(t) = \frac{dG}{dt}, \quad G^*(t) = \int_0^t G(u)du$$

for $t = -2(.1)2$. Graphs are given of $G(t)$, and also of the curve represented parametrically by

$$x = G(t), \quad y = \left[\frac{\alpha}{2\pi G'(t)}\right]^{\frac{1}{2}} \exp\{\alpha[G^*(t) - tG(t)]\}$$

for $\alpha = 1$ and $\alpha = .01$.

A. E.

1233[L].—E. F. M. VAN DER HELD, "The contribution of radiation to the conduction of heat. II. Boundary conditions," *Appl. Sci. Research* A, v. 4, 1953, p. 77–99.

The Appendix (p. 92–99) contains a collection of formulas and some numerical tables concerning the function

$$K_n(x) = \int_1^\infty t^{-n} e^{-xt} dt$$

which is closely related to the complementary incomplete gamma function.

Table VII gives values of K_1, K_2, $2K_3$, $3K_4$, $4K_5$ to 4D or 4S for $x = 0(.01).2(.02)2(.5)5(1)8$.

Table VIII gives values to 2 to 4S of

$$\int_1^\infty t^{-1} e^{-xt} \log t \, dt, \quad \int_0^x K_n(x - t) K_2(t) dt, \quad n = 1(1)5$$

for $x = 0, .01, .05, .1, .25(.25)1(.5)5(1)8$.

Table IX gives values to 2 to 4S of

$$\int_0^x K_n(D - x + t) K_2(t) dt, \quad \int_0^\infty K_n(x + t) K_2(t) dt, \quad D = 2, 4, \quad n = 2(1)5$$

in varying ranges, and at varying intervals, of x.

V. KOURGANOFF (see *MTAC*, v. 3, p. 307, RMT **569**) has tabulated K_2, $2K_3$, $\frac{4}{3} - 2K_4$, $\frac{8}{3}x + 4K_5$ for $x = 0(.02)2$. Earlier tables of $K_n(x)$ are listed in FMR *Index*, sec. 14.83.

A. E.

1234[S].—OTTO EMERSLEBEN, "Das elektrostatische Selbstpotential äquidistanter Ladungen auf einer Kreislinie," *Math. Nachr.*, v. 10, 1953, p. 135–167.

If $2n$ charged particles are equally spaced on a circle of radius r with a charge $(-1)^k$ at P_k, $1 \leq k \leq 2n$, then the self-energy of the system is given by the finite sum

$${}_{2n}^{0}\Phi = \frac{n}{r} \sum_{k=1}^{(n)} (-)^k \csc(\tfrac{1}{2}\pi k/n),$$

where (n) indicates that the last term must be multiplied by $\frac{1}{2}$. In a unit lattice $n = \pi r$ and the power series for the cosecant leads to the decomposition

$${}_{n}^{0}\Phi = {}_{n}\bar{\Phi}_0 + {}_{n}\bar{\Phi}_1,$$

where ${}_{n}\bar{\Phi}_0$ comes from the first term of the power series and is similar to the potential of a linear configuration. Both expressions ${}_{n}\bar{\Phi}_0$ and ${}_{n}\bar{\Phi}_1$ involve

the sums $\sum_{k=1}^{(n)} (-)^k k^{-s}$ for various integer values of s and their asymptotic behavior for large n is studied. The term ${}_n\bar{\Phi}_1$ arises from the curvature of the circle and is $O(n^{-1})$ as $n \to \infty$, while ${}_n\bar{\Phi}_0 = -(2 \log 2)n + O(n^{-1})$. The O-term of ${}_n\bar{\Phi}_0$ is also analyzed in somewhat more detail and a table of ${}_n\bar{\Phi}_0$ is given for $n = 1(1)12$.

Similar results are obtained for the problem of n equally charged and equally spaced particles on a circle. The self-energy in this case is

$$ {}_n^0\Phi = \frac{n}{4r} \sum_{k=1}^{n-1} \csc(\pi k/n), $$

and a table is given for $n = 1(1)30$ when $n = 2\pi r$.

The alternating sums $\sum_{k=1}^{(n)} (-1)^k k^m$ reduce to polynomials in n with rational coefficients when m is an integer. These are given explicitly for $m = 0(1)7$. Finally, the graphs of the seven sums $\sum_{k=1}^{(n)} (-1)^k k^s$, $n = 1(1)7$, are plotted as functions of s for $-2.6 \leq s \leq 3$.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **1205.**

239.—NBSCL, *Tables of* 10^x. (*Antilogarithms to the Base* 10.) Applied Math. Ser. No. 27. Washington, 1953.

In working with 10-figure logarithms recently, I noticed a discrepancy between a value given in this new table of 10^x, and a value from GUILLEMIN.[1] Since Guillemin gives 13 figures for 10^x for $x = 0(.0001).6999$, and 12 figures for $x = .6999(.0001).9999$, and DEPREZ[2] gives 14 places $1(1)9999$, every tenth value in v. 27 may easily be checked directly from these tables.

In checking every tenth value for the block from $x = .40000$ to .50000, I find 128 values in error in the last figure; 127 values should have the last figure raised by one, and one value, that for .49270, should have the last figure reduced by one.

It seems rather surprising that a corps of experienced computers, working with the most modern calculating instruments, should spend the amount of time and labor represented by this table simply to smooth out a 200-year-old table without even checking the accuracy of that table. Only the expenditure of a very little extra time would have been necessary to use the Sundstrand machine to subtabulate Deprez's table to tenths and thus to get a 14-place table of 10^x, which, if rounded to ten places would yield a table far superior to Dodson. Reference is made in the introduction of this volume to the Deprez table, so it is certainly known and available to the personnel of the laboratory.

In line with this reasoning, by using the Deprez values for the block .4680 to .4700 in conjunction with COMRIE's simple, accurate, and very

powerful system of end-figure interpolation to tenths (*Supplement to Nautical Almanac*, London, 1931), together with an old Ellis two-register machine, I reproduced the block .46800 to .47000 (p. 236) to 14 places. Rounding this to ten places and comparing page 236 with these values, I find 23 further errors in the 200 entries, besides the four errors already found in this block. Judging from this sample and from the test above, it would appear that about 13 per cent of the entries in Table I are in error by being one too low in the last digit.

This same volume contains a very fine table, Table II, which gives 16-place values for 10^x for $x = 0(.001).999$. Every 100th value in Table I may be checked easily by direct comparison with the second column of Table II. In the block .40000 to .50000 there are eleven of this type of entry in Table I in error, as may be directly ascertained by checking in Table II.

It is to be hoped that in the near future the Laboratory will issue a corrected version of this Table.

A list of the errors referred to above is appended herewith.

The final digit in 10^x is too low by a unit for the following values of x.

40020	41690	43920	46070	46916	48190
40100	780	980	080	921	260
40190	830	990	180	928	270
320	900	44010	240	930	380
390	930	090	700	936	390
440	950	140	710	938	520
450	42110	180	750	944	570
570	160	300	780	963	670
590	240	400	800	965	730
620	260	450	804	990	750
700	520	540	825	47100	810
790	740	610	832	160	910
840	810	620	834	170	920
41220	900	680	837	190	49050
300	43020	690	838	250	180
350	050	740	852	550	530
500	070	780	855	580	650
510	160	800	860	640	750
520	350	45110	863	750	780
530	480	190	867	790	840
550	530	260	868	850	870
570	670	270	903	870	880
590	730	360	908	940	930
620	840	490	909	950	980
640	910	46030	912	48140	990

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[1] A. GUILLEMIN, *Tables de Logarithmes à 3 quatrades et Nombres Correspondants avec 12–13 chiffres*. Paris, 1912.

[2] F. DEPREZ, *Tables for Calculating, by Machine, Logarithms to 13 Places of Decimals*. Berne, 1939.

UNPUBLISHED MATHEMATICAL TABLES

188[D,F].—D. H. LEHMER, *Roots of Unity and Cyclotomic Periods*. Tabulated from punched cards and deposited in the UMT FILE.

In UMT **145** [*MTAC*, v. 6, p. 102] we gave values of the cyclotomic cosines $2\cos 2\pi k/p$ for $k = 1(1)p - 1$, for all primes $p < 97$ to 20D. We now have a companion table of the cyclotomic sines for the same values of k and p to 12D, and therefore a table of the exponential $\exp(2\pi ik/p)$ for these values of k and p or in other words of the primitive p-th roots of unity for p a prime <100 to 12D.

If we write $p = ef + 1$, then the roots of unity may be summed in e "periods" of f roots each according as the index of k with respect to a primitive root g [these primitive roots were chosen as in JACOBI[1]] is congruent to $0, 1, 2, \cdots, e - 1 \pmod e$. These periods η_s $(s = 0, 1, \cdots, e - 1)$ are roots of period equations of degree e with integer coefficients, whose coefficients, discriminant and sum of whose powers are of interest in the theory of numbers. The present table tabulates η_s for $s = 0(1)e - 1$, $e = 5(1)16, 18(2)26, 32$ for all primes of the form $p = ef + 1 < 100$.

D. H. L.

[1] C. G. J. JACOBI, *Canon Arithmeticus*. Berlin, 1839.

189[F].—A. FERRIER, *Étude de* $5n^2 - 1$ *et* $5n^4 - 1$. 35 typewritten pages on deposit in UMT FILE.

This study is in two parts. The first part gives the solutions n of the congruences

$$5n^2 \equiv 1 \pmod P \qquad P = 10\chi \pm 1 < 11{,}370$$

and

$$5n^4 \equiv 1 \pmod P \qquad P = 10\chi \pm 1 < 10{,}000$$

The second gives the factorizations of $5n^2 - 1$ and $5n^4 - 1$ for $n < 5000$ and $n < 1000$ respectively. All primes q above 10^7 which appear have been checked by showing that q divides $2^q - 2$. This was done on the EDSAC with the cooperation of J. C. P. MILLER.

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190[F].—A. GLODEN, *Factorization of* $N^8 + 1$ *for isolated values of N between* 500 *and* 4000. Three typewritten pages on deposit in the UMT FILE.

This table is an extension of a table of CUNNINGHAM[1] for $N < 200$ and a table by the author for $N \leq 500$ [*MTAC*, v. 7, p. 34, UMT **153**]. In the present table 53 complete factorizations of $N^8 + 1$ which are all new with the exception of $N = 512$ and $N = 1024$.

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[1] A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1. London 1923, p. 140–141.

AUTOMATIC COMPUTING MACHINERY

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TECHNICAL DEVELOPMENTS

A Punched Card Machine Especially Equipped for Scientific Computations

The author believes that a description of the punched-card machine now in use at the Norwegian Defence Research Establishment might be of some general interest.

The nucleus of the installation is a type C3 multiplying punched-card machine, purchased from the French Compagnie des Machines Bull. The original plugboard supplied at the time of purchase has been replaced by another almost double the size, and, by making certain other additions, the range of usefulness of the machine has been extended to include many types of calculations which are not normally practicable with punched-card machines.

The machine has 96 counting wheels, which are connected in groups of three by carry-over. Connection of the groups is determined by the plugging of the carry-over. Four 3-digit counters may be used for a multiplier and four others for a multiplicand, thus allowing two 12-digit numbers to be multiplied. The product is formed by the usual left and right hand principle, the two partial products being finally added together. It will be seen that even when two 12-digit numbers are multiplied, there are eight 3-digit counters which are not used. Although the counters do not permit subtractions to be performed directly, a number may be subtracted by adding its nines complement, and using "end-around" carry.

The operations of multiplication, addition, subtraction, conversion of a number into its nines complement and clearing may be performed. In any cycle only one type of operation can be performed. In order to transfer a number from one counter group to another, 48 change-over relays are provided, each relay having three sets of contacts.

A programme is limited by the 44 cycles which are built into the machine. However, the cycles may be connected in any desired sequence and used over and over again. With each cycle is associated a total of 4 "command" hubs, which are connected to the positive side of the 48-volt supply during the time the cycle is in progress. The operation to be performed during a cycle is determined by plugging from the hubs, the change-over relays being energized in the same way.

It is of interest that a multiplication must be completely programmed, one cycle for each digit in the multiplier and one for the addition of the partial products. Two command hubs are used for each digit, one determining the digit selected from the multiplier and the other determining the amount of left shift of the partial product. Furthermore, any digit or digits

of the multiplicand may be suppressed during multiplication with any digit of the multiplier. This feature permits several trick-pluggings, e.g., several products may be formed at the same time, and it may even be possible to obtain the sum of several products in one operation.

In the event that more than one multiplication occurs during a problem, a uniselector is provided. The uniselector is stepped during each multiplication, usually in the cycle where the partial products are added. The impulse which starts the cycle following a multiplication is led to the input of the uniselector, and the cycle which is required to follow the multiplication is then initiated from the output hubs of the uniselector. These hubs, 50 in number, may be used for a variety of other purposes as well.

Eight sign-control units have been connected to eight experimentally selected counting wheels, those which experience in use of the machine have appeared to be most useful for this purpose. Each unit has a change-over relay, six relays having two sets of contacts, and the remaining two having three sets of contacts. The sign control takes up one position if the associated counting wheel indicates a nine and assumes the alternative position when any other number is indicated.

By means of a comparison unit, provision is made for comparison at the upper and lower brushes of two 6-digit numbers from two consecutive cards. Mention must also be made of ten relay units which can remember a pulse. Six of these units are cleared at the beginning of a card cycle, and the other four by impulses sent to the clearing hubs either by x punches or in any cycle by pulses from the command hubs. In addition, there have been installed eight units similar to the first six, except that they clear in the card cycle when the zeros are passing the brushes. If these units are set up from the lower brushes they work exactly as the other six, but by setting them up from the upper brushes other obvious effects are obtained. The outputs from the units consist of a positive voltage on an output hub which is normally connected to the bobbin of one of the change-over relays. The sign-control relays, and the relays controlled by memory units, may be used to switch the succession of cycles. Thus, although the machine is not directly designed for division, this operation can easily be programmed.

Numbers may be extracted from the machine in three different ways. They can be punched into the cards supplied to the machine, or into new cards passing through an auxiliary punch, or they can be recorded by teleprinter. In the first instance the machine must stop, but this may be avoided in the other two cases by inserting a 24-digit relay-storage unit.

In order to check the complicated plugging required in some problems, a panel with 300 indicator lamps, to show the state of all important relays, has been added. A special feature permits the running of single cycles, allowing observation of the behaviour of the machine during this period.

Further technical details are likely to be of interest only to those actually using the equipment, and the author presents now an outline of some problems solved.

(a) **Calculation of primes.** This problem was attempted only as a demonstration test. The method of successive division was used. Observation was made after each division to see if the remainder was zero or not. Division in each case was performed by subtracting the divisor, 10 times the divisor, 100 times the divisor, etc., until the dividend became negative; division

proper was then started from this point. In this way a minimum number of cycles was used. The machine ran continuously for 30 hours, and produced the primes up to 3600 on the teleprinter output. A single punched card was used, giving 7 as the first number to be tested.

(b) **Calculation of ballistic curves.** The ballistic curves for a slow rocket projectile were obtained from

$$\dot{\xi} = -k v \xi$$
$$\dot{\eta} = -k v \eta - g$$
$$\dot{x} = \xi$$
$$\dot{y} = \eta$$
$$v^2 = \xi^2 + \eta^2.$$

In the case considered $kv = a + b(\xi^2 + \eta^2)$ and the simple integration formula

$$\xi_1 = \xi_{-1} + 2h\xi_0$$

could be used.

The initial values of the first two instants and the time of flight were punched into two cards for each curve. The values of t, ξ, η, x and y at second intervals during the total time of flight of 20 seconds were given in tabular form by the teleprinter. Eighty curves were computed in a little over six hours.

(c) **Solution of linear equations.** The solution was obtained by Verzuh's method, (see *MTAC*, v. 3, p. 453–462), but, because of the extra facilities available, the process was considerably simplified.

The first equation was divided by its leading coefficient, and two cards were punched by the auxiliary punch for each coefficient. By means of a sorter these cards were then mixed with the cards of the other equations, and one card was punched for each of the coefficients in the resulting equations, where one unknown was eliminated. The reason for computing two cards in the first case was that, whilst one set of cards was used by the machine for eliminating an unknown from one equation, the other set could be mixed with the cards giving the next unknown. Normally several equations are solved simultaneously. Whenever the machine has finished the calculations for one equation, the teleprinter prints the sum of the cards just punched. As there is an auxiliary card for each equation, containing minus the sum of all the other coefficients in the equation, the teleprinter would always print 0 or -1 if no round-off errors existed. Each equation was preceded by an X-punched master card, different for the first equation and the other equations. This card not only caused the teleprinter to print the contents of the counter giving the sum of the previously punched cards, but also cleared this counter and switched the machine to the correct programme. No manual switching was therefore required. Ten sets of four equations with four unknowns were solved in two hours by this method.

The author believes that it would not be worth while to construct a more flexible electromechanical computor than the one described, because of the inherent slow speed of such punched-card machines. The introduction of electronic techniques into the conventional punched-card system is a stage which will probably be superseded very soon by the use of a true all-purpose electronic computor. Such a machine may, if so desired, have punched cards

as input and output media. For this reason the author has not incorporated an electronic multiplier in the machine described. The cost of the equipment is about 20,000 dollars, and it is already possible to buy an electronic general-purpose computor for about this price. It would seem therefore that the era in which punched-card machines have been applied to scientific computations will very soon be past.

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BIBLIOGRAPHY OF CODING PROCEDURE

The material described below is among that which has been added to the collection at the National Bureau of Standards Computation Laboratory. A similar collection is available at the Numerical Analysis Research, Univ. of California, Los Angeles 24.

Material for inclusion in these collections should be sent to the Computation Laboratory, National Bureau of Standards, Washington 25, D. C., for the attention of J. H. WEGSTEIN.

56. J. W. BACKUS, "The IBM 701 Speedcoding system," Assoc. Computing Machinery, *Jn.*, v. 1, 1954, p. 4–6. (Cf. item **58** of this Bibliography.)

57. H. H. GOLDSTINE, "Some experience in coding and programming with the Institute Computer," Argonne National Laboratory, *Proceedings of a symposium on large scale digital computing machines*, August 3–5, 1953, ANL Report 5181, p. 273–278.

It is pointed out that it is not possible, with present day equipment, to divorce the engineering and mathematical difficulties in program preparation. Limitations on read-around ratio imply restrictions on coding; on the other hand, mathematically redundant checks are useful both in code checking and problem running.

A code-checking scheme recommended is first the checking of the arithmetics of the code (e.g. sub-routines purporting to represent a function accurately or approximately) and then the logics of the code. In the first step a specialization of parameters is often helpful (too great specialization must be avoided—cf. R. A. BROOKER, S. GILL, D. J. WHEELER, *MTAC*, v. 6, p. 112–113). In the second stage it is useful to record the path of the control through the sub-routine complex. Fast printout of the full memory, to be contemplated at leisure, away from the console, has proved advantageous.

The importance of keeping an adequate diary describing the day-to-day development of the solution of a problem is stressed.

It is interesting to note that it has been decided to add hardware to the machine to detect overflows or spills.

J. T.

NBSCL

58. INTERNATIONAL BUSINESS MACHINES CORP., *Bibliography on the Use of IBM Machines in Science, Statistics, and Education*. New York, January 1954.

59. W. L. VAN DER POEL, "Dead programmes for a magnetic drum automatic computer," *Applied Sci. Research*, v. 3B, 1953, p. 190-198.

Dead programming here refers to programs stored in permanent memory and this paper points out the value of such programming.

60. C. Ross, C. S. Fluke, H. S. Toney, *Handbook for OARAC Programming and Coding*. Wright Air Development Center, Nov. 1952.

This is a concise set of notes for guidance and reference in programming for the OARAC. It includes the order code, a glossary of terms, and examples of simple codes.

61. *Programming Univac fac-tronic Systems. Instruction Manual* I. Remington Rand Inc., Philadelphia, 1953, 249 pages.

The Remington Rand Organization is to be thanked for, as well as congratulated upon, the first volume of its Instruction Manual (it is, however, bulky and expensive). It not only provides an excellent means for learning to code on the UNIVAC, but also contains enough information of a general nature to benefit all devotees of high-speed computation. The text is exposed clearly enough so that a veritable tyro in the coding field can learn its principles solely from a perusal of this book without any other aid.

The first eight chapters, which contain the essential information about coding on the UNIVAC, can be mastered in about two weeks by a beginner. Chapter 9, which gives an elementary description of the engineering principles involved in the operation of a high-speed computer, will prove quite interesting for many readers, although it is not essential for the acquisition of coding skill. This reviewer is tempted to make the same statement anent Chapter 10, dealing with the Flow Chart (since she has never found it necessary to use one), but it is sure to arouse the ire of the numerous advocates of this supposed aid in programming.

The last Chapter serves as an Appendix and will be found most useful by the coder, especially as it lists all of the codes in alphabetical order along with complete explanations, whereas in the previous chapters they were given in small sets only and grouped according to operation. The textbook is further enhanced by the inclusion of many exercises in each chapter and of several excellent charts—some bound into the volume, others placed loose within a pocket on the inner back cover.

The beginner will regret the absence of answers to the exercises with which he might compare the results of his first timid attempts at coding. He will welcome, however, the second volume dealing with advanced techniques in programming and coding, promised by the publisher. This reviewer would welcome even more a detailed operators' manual, so that an advanced programmer could learn to use the machine without having to rely on the services of another person.

A word of advice to a newly established laboratory planning to make this manual its sole textbook: One cannot expect a manufacturer to highlight those peculiarities of his machine which for the unwary user become costly and irritable pitfalls—although the UNIVAC is far less vulnerable on this score than are most of the other computers. It is imperative, therefore, to append another chapter to the manual, expounding the danger points and means for avoiding trouble. One illustration should suffice. The UNIVAC command for shifting to the right is

a negative number—a fact which must be borne firmly in mind when attempting to modify this command.

I. RHODES

NBS

62. BETZ, B. K., *An ORDVAC Floating Binary System*. Ballistic Research Laboratories, Aberdeen Proving Ground.
Memorandum Report No. 733.

63. NAVAL PROVING GROUND, Dahlgren, Va. *Routines for the MARK III Electronic Calculator*.

64. MASSACHUSETTS INSTITUTE OF TECHNOLOGY. *Mechanical Translation. Devoted to the Translation of Languages with the Aid of Machines*. Vol. 1, No. 1, March 1954.
The editors of this publication plan to publish it only occasionally and at irregular intervals in 1954. The first issue is devoted primarily to a bibliography of existing literature on the subject.

65. ROY GOLDFINGER, *New York University Compiler System*. New York University, A.E.C. Computing Facility. Feb. 10, 1954.

66. DIGITAL COMPUTER LABORATORY, University of Illinois. *ILLIAC Programming, A Guide to the Preparation of Problems for Solution by the University of Illinois Digital Computer*.
The ILLIAC is available for use by personnel of the University of Illinois provided users code their own problems. This manual is designed to teach how to code for the ILLIAC and how to make use of all its facilities. It includes chapters on: the arithmetic of the ILLIAC, the order code, subroutines and interpretive routines, the decimal order input, scaling, machine methods and coding tricks, checking methods, tape preparation, calculation of running time, preparation of a complete program, the cathode ray tube display, the program library, and definition of terms.
This, like the UNIVAC *Instruction Manual* and WHIRLWIND *Comprehensive System Manual*, is a good example of a text for a course in machine programming or a manual for self taught users of a specific electronic digital computer.

J. WEGSTEIN

NBSCL

67. DIGITAL COMPUTER LABORATORY, Massachusetts Institute of Technology. *WHIRLWIND Memoranda* M-2527, M-2539.
The WHIRLWIND, operating in a supervisory mode, is used to simulate different easy-to-use digital computers. One of these pseudo-computers is called the "CS Computer" (Comprehensive System) and another is called the "Summer Session Computer." These memoranda include manuals which are suitable as texts for a course in programming for the respective computers. These memoranda also provide a description of other features of the very interesting experiments in systems of high speed computation being conducted at M.I.T.

BIBLIOGRAPHY Z

1133. ANON., "New digital computers developed," *Aviation Week*, April 12, 1954, p. 60–63.

A brief account of the new Bendix Computer and ElectroData Corp.'s Model 203 (ElectroData Corp.—a subsidiary of Consolidated Engineering Corp.).

The Bendix Computer is a digital differential analyzer, operating in a decimal numbers system. It can be used to solve linear and non-linear differential equations or simultaneous sets of such equations, or for integral equations, split-boundary value problems, and individual or simultaneous sets of linear or nonlinear algebraic and transcendental equations. The machine can be programmed from punched tape or a manual control panel; output is recorded on punch tape.

The Model 203 is a high speed general purpose computer, operating serial fashion in binary coded decimal number system. It includes a magnetic drum as well as an accessory magnetic tape storage unit. Its input may be from standard punch-card equipment or from a specially designed photoelectric reader operating from previously punched tape. Output can be recorded on standard punch cards, tape, or be tabulated on an electric typewriter.

Both computers as described by their manufacturers are in the medium price range.

A. R. COCK

NBSCL

1134. ANON., "These machines think," *Signal*, Nov.–Dec. 1953, p. 14–16.

This article describes the first calculator of comparable capacity to be produced in quantity—the IBM Data Processing Machine, the 701 which rents for $11,900 a month, or more, depending on storage capacity.

Its memory devices consist of cathode ray tubes, magnetic drums and magnetic tapes. It is capable of performing more than 16,000 addition or subtraction operations a second and more than 2000 multiplication or division operations a second. The magnetic Drum Storage Units will store more than 80,000 digits and the Magnetic Tapes have a capacity of more than 2,000,000 digits a tape. Results can be printed by means of a 150 line-a-minute printer at the rate of 1050 ten-digit numbers a minute or punched on cards at the rate of 2400 ten-digit numbers a minute or stored on magnetic tapes at the rate of 1200 ten-digit numbers a second.

R. K. ANDERSON

NBSCL

1135. B. V. BOWDEN, ed., *Faster than thought, a symposium on digital computing machines*. London, Pitman and Sons, Ltd., 1953, 414 p. Price 35 shillings.

Faster than thought is an interesting account of the development and application of electronic digital computing machines in England. In this symposium, Dr. BOWDEN has edited the contributions of twenty-four British experts, including himself, to produce an entertaining and satisfying description of the genesis of modern digital computing machines, their theory

and design and their multi-faceted applications in industry, commerce and scientific research.

Beginning with a foreword by the Earl of HALSBURY, managing director of the National Research Development Corporation and a preface by the editor, the book is divided into three main parts: Part One, The History and Theory of Computing Machines; Part Two, Electronic Computing Machines in Britain and America; Part Three, Applications of Electronic Computing Machines. Included also are a brief index, glossary and two appendices, one a reprint of pages 666–731 of TAYLOR's *Scientific Memoirs*, v. III, including editorial notes by the translator, the Countess of LOVELACE, the other extracts from four letters from the *Lovelace Papers*.

Some indication of the balance and flavour of the book can be gleaned from the enumeration of the contents of its twenty-six chapters, five of which are included in Part One, nine in Part Two and the remaining twelve in Part Three. Part One includes the following: A Brief History of Computation, The Circuit Components of Digital Computers, The Organization of a Typical Machine, The Construction, Performance and Maintenance of Digital Computers, Programming for High-speed Digital Calculating Machines. Part Two treats the University of Manchester Computing Machine, calculating machine development at Cambridge, automatic computation at the National Physical Laboratory, the Harwell electronic digital computer, the Telecommunications Research Establishment parallel electronic digital computer, the Imperial College Computing engine, the Royal Aircraft Establishment sequence-controlled calculator, calculating machines at the Birbeck College Computation Laboratory and six pages on computers in America. In Part Three, special-purpose automatic computers are defined and there are discussed, in separate chapters, applications of electronic computing machines in logical problems, crystallography, meterology, ballistics engineering, government calculations, business and commerce, economics, dynamical astronomy, the playing of games—nim, draughts and chess. Finally, in the closing chapter there is a stimulatingly temperate comparison of thought and machine processes.

With respect to *Faster than thought*, the reviewer is inclined to agree with the statement in the foreword that "It is undoubtedly the best general account yet written." It contains much material which will interest both specialists in the development of electronic computing machinery and potential users of this amazing equipment. Its account of the work of CHARLES BABBAGE a hundred years ago, the interest of Lady LOVELACE in his attempts to build large-scale automatic digital machinery and her pioneering work in programming, the competent discussion of computer applications ranging from playing chess or draughts to payroll computations and scientific calculations—all these are among many features of the book which will make it stimulating reading for the general reader and specialist alike.

Accomplishment in the electronic digital computer field has reached a magnitude which makes it impracticable to achieve completeness in a reasonable length exposition on the subject. It nevertheless is difficult to understand the omission of mention of the epoch-making large-scale Bell Telephone Laboratories' Relay Computers and the parallel-mode, electro-

static-store National Bureau of Standards Western Automatic Computer in *Faster than thought*. Again brief mention could have been made of the versatile E. U. CONDON's relay machine to play the game of nim demonstrated about two decades ago at the New York City World's Fair, evoking considerable interest there. Finally, some indication of the trend of activity concerning electromagnetic and electronic digital computers on the Continent, including the development of computers and their application, would not have been unwelcome. In the evaluation of the book as a whole, these points appear of minor consequence and Dr. BOWDEN is to be congratulated upon his contribution to the literature on modern electronic digital computers.

E. W. C.

1136. D. T. CAMINER, L. FANTL, G. S. D. KING, T. R. THOMPSON, & W. B. WRIGHT, "The use of a high speed automatic calculator in the refinement stages of crystal-structure determinations," *Acta Crystallographica*, v. 7, Part 3, 1954.

Refinement can be said to begin when approximate atomic positions have been found, and the phases of the most important structure factors are known. Best-known methods available for refinement are 1. The Fourier Method, 2. The Differential Fourier Method, 3. The $(F_0 - F_c)$ Fourier Method and 4. The Method of Steepest Descent and Least Squares. For automatic calculation, the authors found method (2) most appropriate for the first stages of the refinement of coordinates and method (4) for the final stages. The Quarichi (1949) modification of (4) was used.

The calculator used was the LEO, a machine built for the Lyons Electronic Office. LEO is a serial, single-address, binary machine. It has a store of sixty-four mercury delay lines, providing a capacity of 2048 words of 17 binary digits. It has paper-tape input, photoelectrically read.

Programs were worked out for nicotinamide, which has the space group $p2_1/a$. The computational procedure could be readily modified for other centrosymmetric space groups. Application to acentric space groups would be more complicated.

Flow diagrams are given for the two programs of computation. An analysis is made of storage space required and the time needed for the calculations.

E. W. C.

1137. W. C. CARTER & M. ELLIS, "A comparison of order structures for automatic digital computers," Operations Research Soc. of Amer. *Jn.*, v. 2, no. 1, 1954, p. 41–58.

A method is outlined whereby order structures of the 1-, 2-, 3-, and 4-address types may be analyzed for application in a particular type of computation. The purpose is to assist in determining which will provide the greatest speed of operation and the greatest efficiency of storage in a proposed machine design. The method is applied to a payroll calculation, and the resultant computation times, on comparable machines with single-address and 3-address order structures, are presented. Some details of the

different types of orders assumed to be used with the two different order structures are given.

R. A. KIRSCH

NBSEC

1138. M. J. MENDELSON, "The decimal digital differential analyzer," *Aeronautical Engineering Review*, v. 13, 1954, p. 42–54.

This is a very clear and simple introduction to the logic of the digital differential analyzer. It explains the algorithm for numerical integration, the method of representing negative numbers and zero or negative rates, the organization of digital integrator registers on a magnetic drum, some examples of interconnections to solve simple differential equations, the rules governing scaling of magnitudes, and some special techniques. The latter include the decision integrator, the signum function, constant multipliers, multiplication, and division.

R. D. ELBOURN

NBSEC

NEWS

The five hundred and second meeting of the American Mathematical Society was held at the University of Chicago on Thursday, Friday, and Saturday, April 29, 30 and May 1, 1954.

The program included a symposium in Applied Mathematics (sponsored by the Society and the Office of Ordnance Research) of which the following papers are relevant here.

Session II W. W. LEUTERT, Ballistic Research Laboratories, *Chairman*.

Computational methods, M. R. HESTENES, University of California, Los Angeles

Motivations for working in numerical analysis, JOHN TODD, National Bureau of Standards

Some numerical computations in ordnance problems, A. A. BENNETT, Brown University

Western Computer Conference and Exhibit. The Western Computer Conference on "Trends in Computers: Automatic Control and Data Processing" was held at the Ambassador Hotel in Los Angeles, California, February 11–12, 1954, under the joint sponsorship of the American Institute of Electrical Engineers, the Institute of Radio Engineers and the Association for Computing Machinery.

Opening Ceremonies—Thursday, February 11, 10:30 AM

Welcome Address	D. H. LEHMER, Professor of Mathematics, University of California, Berkeley
Keynote Address	W. W. MCDOWELL, Director of Engineering, International Business Machines Corp., Endicott, N. Y.

Session I—Thursday, February 11, 2:00 PM

Automatic Control—Systems	JOHN M. SALZER, Hughes Research and Development Laboratories, Culver City, California, *Chairman*
An Experimental Digital Flight Control System	MAIER MARGOLIS, J. B. Rea Company, Inc. and ERIC WEISS, Dynalysis, Inc.
The DIGITAC Airborne Control System	E. E. BOLLES, D. W. BURBECK, W. E. FRADY, E. M. GRABBE, Hughes Research and Development Laboratories

Application of Operational Digital Techniques to Industrial Control	BERNARD M. GORDON, Laboratory for Electronics, Inc.
Digital-Analog Machine Tool Control System	HARRY W. MERGLER, Lewis Flight Propulsion Laboratory of the National Advisory Committee for Aeronautics, Cleveland, Ohio
Experiments with a Digital Computer in a Simple Control System	T. J. BURNS, J. D. CLOUD, J. M. SALZER, Hughes Research and Development Laboratories

Session II—Thursday, February 11, 2:00 PM

Data Processing—Systems	RICHARD G. CANNING, University of California at Los Angeles, *Chairman*
The Automatic Handling of Business Data	OLIVER WHITBY, Stanford Research Institute
Business Data Processing: A Case Study	
A. The Retailer's Unit Control Problem	SAMUEL J. SHAFFER, The May Company, Los Angeles
B. Unit Control Systems Engineering	RAYMOND DAVIS, Librascope, Inc., Glendale, California
C. A Solution for Automatic Unit Control	HARRY D. HUSKEY, Institute for Numerical Analysis
D. The System in Operation	MYRON J. MENDELSON, Computer Research Corporation

Session III—Friday, February 12, 2:00 PM

Automatic Control—Equipment	ROGER SISSON, Computer Research Corporation, Hawthorne, California, *Chairman*
Approaches to Design Problems in Conversion Equipment	A. K. SUSSKIND, Servomechanism Laboratory, Massachusetts Institute of Technology
A Multi-Channel Analog-Digital Conversion System for D-C Voltages	W. S. SHOCKENCY, Hughes Research and Development Laboratories
A High-Speed Multichannel Analog-Digital Converter	JAMES M. MITCHELL, J. B. Rea Company, Inc.
A Shaft-to-Digital Encoder	B. M. GORDON, M. A. MEYER, R. N. NICOLA, Laboratory for Electronics, Inc., Boston, Massachusetts
Real-Time Digital Differential Analyzer ("Dart")	LOREN P. MEISSNER, U. S. Naval Ordnance Laboratory, Corona, California

Session IV—Friday, February 12, 2:00 PM

Data Processing—Equipment	HARRY E. BURKE, JR., Consolidated Engineering Corporation, Pasadena, California, *Chairman*
The IBM Magnetic Drum Calculator Type 650-Engineering and Design Considerations	E. S. HUGHES, JR., Endicott Laboratory International Business Machines Corp., Endicott, New York
Remington Rand Speed Tally—Design Features	J. L. HILL, Engineering Research Associates, St. Paul, Minnesota
Production Control with the Elecom 125—Electronic Business System	NORMAN GRIESER, Underwood Corporation, Long Island City, New York

A Centralized Data Processing Center	J. J. DOVER, Air Force Flight Test Center, Edwards Air Force Base, Edwards, California
A Merchandise Control System	WILLIAM L. MARTIN, Telecomputing Corporation, Burbank, California

Discussion groups Friday morning, February 12, included the following topics:

Unit Control in Retail Operations
Numerical Control of Petroleum and Chemical Processes
Numerical Control of Machine Tools
Maintenance Requirements for Business Computers
Mathematical Methods in Management Programming

Symposium on Automatic Programming for Digital Computers. Sponsored by the Navy Mathematical Computing Advisory Panel, a symposium was held in Washington, D. C. on Thursday and Friday, May 13 and 14, 1954.

May 13, Morning Session: C. V. L. SMITH, ONR, Washington, *Chairman*
- *Definitions*, GRACE HOPPER, Remington Rand Inc., Philadelphia
- *Differentiators*, HARRY KAHRIMANIAN, Naval Aviation Supply Office, Philadelphia
- *Compiler Method of Automatic Programming*, NORA MOSER, Army Map Service, Washington
- *Editing Generators*, JOHN WAITE, RCA Victor, Camden
- *New York University Compiler System*, ROY GOLDFINGER, NYU
- *Application of Automatic Coding to Logical Processes*, BETTY HOLBERTON, David Taylor Model Basin
- Discussion

May 13, Afternoon Session: ALBERT E. SMITH, BuShips, Washington, *Chairman*
- *The MIT Systems: Comprehensive, Summer Session, and Algebraic*, C. W. ADAMS and J. H. LANING, JR., MIT
- *Interpretive Routines in the Illiac Library*, DAVID E. MULLER, Univ. of Ill.
- *Planning Universal Semi-Automatic Coding*, SAUL GORN, Ballistics Research Lab., Aberdeen Proving Ground
- *Present Status of the Michigan Automatic General Integrated Computation (MAGIC)*, J. H. BROWN and J. W. CARR III, Univ. of Mich.
- Discussion

May 14, Morning Session: S. B. WILLIAMS, Consulting Electrical Engineer, Washington, *Chairman*
- *Automatic Programming on the Burroughs Computer*, HUBERT M. LIVINGSTON, Burroughs Corporation
- *IBM 701 Speed Coding System*, JOHN W. BACKUS and HARLAN HERRICK, International Business Machines Inc., New York
- *Programming for the IBM 701 with Repetitively Used Functions*, ALLEN KELLER and R. A. BUTTERWORTH, General Electric Company, Lynn, Mass.
- Discussion
- Summary and Forecast, GRACE HOPPER, Remington Rand Inc.

National Applied Mathematics Laboratories. The National Bureau of Standards Institute for Numerical Analysis, operated on the Los Angeles Campus of the University of California since 1947, ceased to exist on June 30, 1954. Most of its activities will be continued under the sponsorship of the Office of Naval Research and the Office of Ordnance Research under the administration of the Department of Mathematics of the University of California at Los Angeles of which M. R. HESTENES is chairman; C. B. TOMPKINS will be the Director of the new unit which will be called Numerical Analysis Research, [NAR]. The physical facilities of INA, including the SWAC and the library, will be loaned by the National Bureau of Standards to the University of California.

A new section, Numerical Analysis, has been established July 1, 1954 by the National Bureau of Standards in Washington; at the same time, the Machine Development Laboratory has been replaced by a section Mathematical Physics.

The new organization of the National Applied Mathematics Laboratories will be as follows:

F. L. ALT, Acting Chief, E. W. CANNON, Assistant Chief

1. Numerical Analysis, J. TODD, Chief
2. Computation Laboratory, M. ABRAMOWITZ, Acting Chief
3. Statistical Engineering Laboratory, C. EISENHART, Chief
4. Mathematical Physics, E. W. CANNON, Chief

OTHER AIDS TO COMPUTATION

BIBLIOGRAPHY Z

1139. R. R. BENNETT, "Analog computing applied to noise studies," I. R. E., *Proc.*, v. 41, 1953, p. 1509–1513.

A problem of great importance in modern communication engineering is the analysis of noise in linear and nonlinear systems. For a linear system the weighting function (impulsive response, Green's function) completely characterizes the system and is a necessary tool in spectrum shaping analysis. The author discusses the analog simulation of the weighting function using the concept of the adjoint system (cf. H. H. LANNING, JR. & R. H. BATTIN, *MTAC*, v. 7, 1953, p. 125). For nonlinear systems the author briefly describes how the amplitude distribution may be shaped using a function generator. Finally, statistical methods—using the concept of confidence levels—are discussed for determining the accuracy, say of the standard deviation, of a number of identical runs on the same physical system.

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1140. M. B. COYLE, "An airflow analogy for the solution of transient heat conduction problem," *British Jn. Applied Physics*, v. 2, 1951, p. 12–17.

The airflow analogy parallels closely the familiar electrical resistance-capacity analogy for transient heat conduction phenomena. Quantity of heat and temperature in the heat conduction system are represented in the airflow system by mass of air and pressure respectively. The continuous thermal system is considered replaced by a lumped system which is simulated by a network of air reservoirs, each containing a mass of air which is a function of the air pressure within it, inter-connected by capillary tubes through which the mass airflow is proportional to the pressure difference between the ends. The air reservoirs are in the form of U-tubes with a confining fluid, all the U-tubes having one leg in common. With this arrangement, the capacity (taken in the electrical sense to mean the mass of air discharged for unit fall in pressure) of a reservoir may be made a function of pressure by constructing it with variable cross-sectional area. Therefore, a problem involving variable thermal properties may be set up with appropriately

constructed reservoirs and a solution obtained with no further manipulation required during the test.

This appears to be an advantage of the air-flow analogy relative to the electrical analogy since in the latter variable thermal properties are simulated by varying circuit elements in the course of a test, either manually or automatically. However, the electrical analogy appears to possess far greater flexibility and inherent experimental accuracy.

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1141. LEONARD J. CRAIG, "The magnetic amplifier as an analog computer component," I. R. E., *Proc.*, v. 41, 1953, p. 1477–1482.

It is shown that the load voltage of a parallel-connected magnetic amplifier is a gate whose width is directly proportional to the control current, when a square wave source voltage is applied. Tests indicate that this relationship holds to within 0.1 percent over a wide range of source voltage (except for scale factor), load resistance, and temperature, provided the magnetic material used has reasonably ideal characteristics.

This phenomenon is utilized in several circuits: a linear pulse width modulator, a linear pulse-position generator and a square-wave phase shifter. The phase shifter was applied to a DC multiplier which achieved product accuracies of 3 percent, without the use of high precision components.

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1142. W. A. MCCOOL, "An AM-FM electronic analog multiplier," I. R. E., *Proc.*, v. 41, 1953, p. 1470–1477.

A multiplier is described wherein an AM signal whose index of modulation is proportional to an input voltage is applied to an attenuator whose gain is proportional to frequency deviation (an FM discriminator). Frequency deviation is made proportional to a second input, so the discriminator output, with the carrier subtracted, is the product of the inputs. Feedback techniques stabilize the AM and FM modulation indices. Although basically a two-quadrant multiplier, it is arranged, by a conventional method, for four-quadrant operation.

A total of 24 tube envelopes is used. Performance is limited primarily by stability of discriminator parameters, and is independent of tube characteristics. A long time multiplication error of less than 1 percent of maximum output is reported. Usable bandwidth (flat amplitude response and less than 15 degree phase shift for highest frequency component) is approximately 0.1 percent of carrier frequency (1 percent was reported, apparently erroneously).

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1143. JAMES C. ROBB, "A calculator for aiding matrix calculations," Faraday Soc. *Trans.*, v. 50, 1954, p. 8–12.

Twelve Curta Calculators are mounted on a base and have their working mechanisms geared together and centrally controlled. The operations (1) shift carriage, (2) clear and (3) enter multiplier, activate the twelve calculators simultaneously. Complete constructional details are given. The capacities are, of course, those of the Curta—6D multiplier, 8D multiplicand and 11D product. Provision for positive and negative multiplier and multiplicand is possible.

The device is well suited to problems involving many simultaneous multiplications and addition. As an example, matrix multiplication is described. The writer claims the equipment is suitable for use with inversion schemes like those of Crout, but no further explanation is given. For twelfth order matrix by vector multiplication with 4D elements, a time of 17 minutes is cited.

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1144. O. J. M. SMITH, "Economic analogs," I. R. E., *Proc.*, v. 41, 1953, p. 1514–1519.

The possibility of using electronic analog computers for problems in economics is discussed. Several circuits are given for specific simple problems and economic terms are interpreted in terms of electrical networks; for example, if the supply (outcome) influences the demand (input) then there is feedback in the system. The author points out that the concepts of closed loop system, Nyquist diagrams and vector diagrams are now being used by economists to interpret and analyze a variety of economic problems. A list of forty references is included.

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1145. G. W. SWENSON & T. J. HIGGINS, "A direct-current network analyzer for solving wave-equation boundary-value problems," *Jn. Appl. Phys.*, v. 23, 1952, p. 126–127.

The authors discuss the solution of the partial differential equation

$$\nabla^2\varphi + (\omega/c)^2\varphi = 0$$

on the function $\varphi(x, y)$ by means of a resistance network. This network is analogous to the usual network for the Laplacian, except that the term $(\omega/c)^2$ requires the introduction of a "negative resistance" involving an amplifier between each node and ground. This "negative resistance" must be readjusted manually by an iterative method in the course of the problem. Both characteristic value problems and forcing function problems are considered.

F. J. M.

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1146. G. H. VINEYARD, "Simulation of trajectories of charge particles in magnetic fields," *Jn. Appl. Phys.*, v. 23, 1952, p. 35–39.

The author sets up an equivalence between the motion of a charged particle in an electric and magnetic field and the motion of a sphere rolling on a rotating surface, subject to gravity and viewed from an independently rotating coordinate system. A variety of interesting special cases corresponding to the betatron and magnetron are discussed. In addition an experimental technique involving a rotating glass surface and multi-exposure photography for obtaining the orbits is described. A preliminary setup was made with immediately available components and the author considered the results satisfactory.

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NOTES

163.—ANOTHER UNRECORDED PITISCUS ITEM. In my article on BARTHOLOMÄUS PITISCUS (1561–1613), to whom our word Trigonometry is due (1595) (*MTAC*, v. 3, p. 390–397, 1949), I referred to one of his tables, "unlisted in any of the ordinary bibliographic or historical sources" (p. 396), entitled

> SINUUM/TANGENTIUM/et/SECANTIUM/Canon Manualis/*Accomodatus ad trigo/nometriam/BARTOLOMAEI*/Pitisci/Grünbergensis Silesij./[Decoration]/HEIDELBERGAE,/*Typis Iohan Lincelloti, Acad Typo.*/Imprensis Ionâe Rosâe./MDCXIII./Signatures A^{12}–H^{12}, I^4 [200 p.].

It contained 5D tables for all six of the trigonometric functions in the quadrant, at interval 1′. Opposite pages record the values for 30 minutes of each function; but both pages are headed "sinus," "tangens," "secans." The angles on the right-hand pages being complementary to those on the left, the values on the right-hand pages are really those of cos, cot, csc for the angles on the left-hand page. Thus 180 pages, A^2 *verso*–H^8 *recto*, are occupied with the tables. "Explicatio numerorum hujus Canonis" occupies pages A^1 *verso* to A^2 *recto*. A^1 *recto* is the title page. H^8 *verso*–H^{12} *recto* are devoted to text explanations; H^{12} *verso* and I^1 *verso* blank; I^1 *recto* comment on the following 230 Errata (sin and cos 46; tan and cot 86; sec and csc 96; 2 others): I^2 *recto*–I^4 *recto*.

The little pages are of size 7.3 × 13.2 cm. A film copy of this table for the Library of Brown University was made from a copy of the original at the University of Illinois.

In the summer of 1953 the bookshop Old Authors Farm, R.R. 1, Harrisburg, Ontario, Canada, offered an extraordinary collection of mint copies of old books—duplicates from the Vatican Library. Among these was a second edition of the Pitiscus volume described above. The displays of the title pages, down through the word HEIDELBERGAE, except for a new decoration, are identical; then follows in the new volume: Typis Joan. Georg. Geyder. Acad. Typ. Imprensis Jonâe Rosâe./MDCXX./A^{12}–H^{12}. This volume, acquired by Mr. ALBERT E. LOWNES of Providence, R. I., further

enhancing the value of his remarkable personal library in the field of history of science, was kindly placed at my disposal for preparing this article.

It will be observed that there are eight fewer pages in the new edition; this was due to elimination of the 230 errata noted in the first edition, and correction of the corresponding places in the earlier pages. I checked the 46 noted errors for sines and cosines and found that while one of those was incorrectly listed, not only were all the others corrected, but also some errors not previously noted. In spot checking of the other functions no error was found in the new edition. Otherwise pages A^2–H^{12} in both editions appear to be textually identical.

The little volume is bound in old vellum, seemingly as fresh as on the day the volume was published 334 years ago—seven years after the death of Pitiscus.

Apart from this copy, the only other one known is in the Vatican Library.

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164.—PERTURBATION OF LAGRANGIAN COEFFICIENTS. Suppose it is desired to interpolate a function $g(x)$ polynomially to a certain value when the independent variable x is a function, $f(t)$, of a parameter, t, where the values g are known at equally spaced values of t. Under certain circumstances we find it advantageous to carry out a first order perturbation of the Lagrangian coefficients for equally spaced values. Suppose $f(t)$ is a function with a non-vanishing derivative and with a continuous third derivative. Also, suppose we want to interpolate $g(x)$ to the value $x = f(a)$, given $f(t)$ for $t = a + (i - b)h$, $i = 0, \cdots, m$ where b is a positive number less than m and h is a positive number. Then, from Taylor's series

$$(1)\quad f(a + (i - b)h) = f(a) + (i - b)hf'(a) + \tfrac{1}{2}(i - b)^2h^2f''(a) + R_i$$

where R_i is the appropriate remainder. For the purpose of interpolation $f(a)$ has no effect on the coefficients. Since the values $(i - b)hf'(a)$, $i = 0, \cdots, m$ are equally spaced, the Lagrangian coefficients for interpolating to x are given by

$$(2)\qquad L_j = \prod_{i \neq j} \frac{(x - x_i)}{(x_j - x_i)} \quad j = 0, 1, \cdots, m.$$

Now, considering the x's as variables we find the logarithmic derivative of L_j to be

$$(3)\qquad \frac{dL_j}{L_j} = \sum_{i \neq j} \frac{dx - dx_i}{x - x_i} - \sum_{i \neq j} \frac{dx_j - dx_i}{x_j - x_i}.$$

Then letting

$$x_i = (i - b)hf'(a), \quad dx_i = \tfrac{1}{2}(i - b)^2h^2f''(a)$$

$$x = 0, \qquad dx = 0$$

$$\frac{dL_j}{L_j} = \frac{hf''(a)}{2f'(a)}\left[\sum_{i \neq j} (i - b) - \sum_{i \neq j} (i + j - 2b)\right]$$

$$= \frac{hf''(a)}{2f'(a)}\left[m(b - j)\right] \quad j = 0, \cdots, m.$$

Hence the perturbed values of the Lagrangian coefficients are given by

$$L_j + dL_j = \prod_{i \neq j} \left(\frac{b-i}{j-i} \right) \left[1 + \frac{hf''(a)}{2f'(a)} m(b-j) \right]. \tag{4}$$

The sum of the perturbed coefficients in (4) equals 1 since

$$\sum_{j=0}^{m} \left[(b-j) \prod_{i \neq j} \left(\frac{b-i}{j-i} \right) \right] = 0.$$

While an error estimate might be calculated, in any application it is simple to calculate actual coefficients to compare with the first order approximations.

For example, let $f(t) = t^3$, $t_i = a + (i - 1.5)h$ to $i = 0, 1, 2, 3$. Then $f''(a) = 6a$, $f'(a) = 3a^2$ and for interpolation to $x = a^3$ from 4

$$L_0 + dL_0 = -\frac{1}{16}\left(1 + \frac{9h}{2a}\right)$$

$$L_1 + dL_1 = \frac{9}{16}\left(1 + \frac{3h}{2a}\right)$$

$$L_2 + dL_2 = \frac{9}{16}\left(1 - \frac{3h}{2a}\right)$$

$$L_3 + dL_3 = -\frac{1}{16}\left(1 - \frac{9h}{2a}\right).$$

Clearly these coefficients should be used only if $\frac{h}{a}$ is small. Formulas of this type when appropriate have been used at the Los Alamos Scientific Laboratory in a calculation on the IBM 701 to save appreciable storage space compared to that required for direct calculation of the correct coefficients. Of course, $f''(a)/f'(a)$ may not be simple to calculate in many applications.

In case it is inconvenient to calculate $f'(a)$ and $f''(a)$ we may interpolate to $x = f(a)$ using the following adjusted coefficients

$$L_j + dL_j = \prod_{i \neq j} \left(\frac{p+b-i}{j-i} \right) \left[1 + \frac{hf''(c)}{2f'(c)} m(p+b-j) \right] \tag{5}$$

$$j = 0, \cdots, m,$$

where c is a convenient value of t, usually taken to be one of the equally spaced values near a and $p = (a - c)/h$. When $a = c$, (5) reduces to (4). The sum of the adjusted coefficients in (5) is 1.

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165.—AN INVESTIGATION INTO THE REAL ROOTS OF CERTAIN POLYNOMIALS. In connection with work on the application of statistical theory to the determination of the number and location of roots of high order poly-

nomial equations, it was required to investigate the real roots of the equations:

$$(1) \qquad x^8 \pm x^7 \pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1 = 0$$

where all possible combinations of sign are taken.

It can be shown simply, that all roots of these equations lie in the range:

$$\tfrac{1}{2} < |x| < 2$$

and the computational problem therefore reduces to that of evaluating each of the 256 different equations obtained by changing the signs of (1), over the range $\frac{1}{2} \leq x \leq 2$, and at a sufficiently fine interval to ensure that no roots are missed.

Preliminary calculations indicated that the functions are quite well behaved, and that tabulation at intervals of $\frac{1}{16}$ in x would be adequate, and it was thus necessary to evaluate each equation at 25 points.

The polynomials were written in the form:

$$x(\cdots x(x(x \pm 1) \pm 1)\cdots) \pm 1$$

and were evaluated by successive addition of the coefficients and multiplication by x.

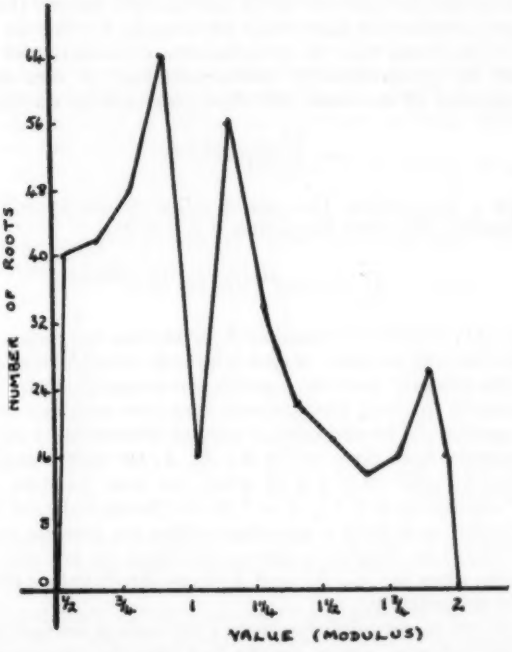

FIG. 1.

The calculations were performed on the HEC 1 and the time for calculation and printing of each polynomial was 50 seconds. For convenience of reference, the sign combination of each polynomial was printed as well.

An analysis of the results of this tabulation shows that of the 256 equations considered:

58 have no real roots
190 have two real roots
8 have four real roots.

The distribution of the roots by numerical value is shown in Fig. 1. The detailed results of this tabulation are available for reference at the Computation Laboratory, Birkbeck College.

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166.—NUMERICAL STUDY OF SIGNATURE RANK OF CUBIC CYCLOTOMIC UNITS. In the search for algebraic fields (particularly beyond the quadratic) where unique factorization fails, little work seems feasible on high speed computers for the reason that the straightforward evaluation of forms must be controlled by group-theoretic data-processing.[1] A rare opportunity, however, is afforded by the cubic field $R(\alpha)$ generated by the Gaussian sum

$$\alpha = \sum_{x=0}^{p-1} \exp 2\pi i x^3/p$$

where $p = 6n + 1$ is a prime. Here the so-called *cyclotomic unit*[2] is given by the simple formula (for three conjugates $0 \leq i \leq 2$):

$$\Theta_i = -\prod_{t=0}^{n-1} \sin\left(\tfrac{1}{2}\pi g^{3t+i}/p\right) \csc\left(\tfrac{1}{2}\pi g^{3t+i+1}/p\right)$$

where g is an odd primitive root modulo p (or an even one augmented by p). Now while the decimal accuracy of such a formula would be lost very rapidly the Θ_i have the property that the possible non-unique factorization in $R(\alpha)$ can be of a certain frequent type (namely *even* class number) only when all three Θ_i are positive.[2] This condition is easy to determine by machine.

We accordingly form three tallies A_0, A_1, A_2 for each p as follows: We increment the A_i tally $(0 \leq i \leq 2)$ when the least positive residues of g^{3t+i+1} and g^{3t} modulo $2p$, $0 \leq t \leq n-1$, lie in different halves of the interval $(0, 2p)$ subdivided at p. Such a procedure tallies the possible negative sign of $\sin\left(\tfrac{1}{2}\pi g^{3t+i+1}/p\right) \csc\left(\tfrac{1}{2}\pi g^{3t}/p\right)$; hence the three Θ_i are seen to be all positive exactly when the A_0, A_1, and A_2 have (final) values that are odd, even, and odd respectively.

Since there are many primitive roots g for each p we find that the A_i depend on the g chosen, except for the fact that the occurrence of three positive Θ_i must be independent[2] of g. Assuming that the powers of g lay

in half-intervals "at random" we should expect the A_i to be each approximately $p/12$, while assuming "random parities," (which is more precarious), we should expect the Θ_i to be all positive approximately $\frac{1}{4}$ of the time; (since $\Theta_0\Theta_1\Theta_2 = +1$, or, equivalently, A_2 is always odd).

The computation was performed on the MIDAC starting Jan. 27, 1954, for the first 207 values of p (<3000). The values[3] of p and g in the form of binary-coded decimals were placed as alternate words on high-speed photoelectric input tape. They were read in by pairs (taking approximately .15 sec. for each pair). The conversion to binary form was internal. For the main induction loop the starting point was each g^{3t}, $0 \leq t \leq n-1$, from which were formed g^{3t+i+1} for $0 \leq i \leq 2$ (always modulo $2p$). The three tally increment decisions (for A_i) were made as described above (taking .04 sec. for each circuit of the loop). In addition, every time the MIDAC chose not to increment an A_i tally it incremented a common D tally. After the loop had been traversed n times two checks on accuracy were made. First the sum of the three A_i and the D tallies was verified to be $(p-1)/2$ (the number of increment decisions), and secondly the last power, $g^{(p-1)/2}$ (reduced modulo $2p$), was verified to be $2p-1$. The MIDAC was made to examine the three final A_i tallies for the occurrence of alternate parities and indicate such occurrences or non-occurrences in the form of a -0 or 0 respectively in a temporary storage space. The output for each 0 consisted of

$$p \; g \; A_0 \; A_1 \; A_2 \; -0 \text{ (or } 0)$$

where the first five items were converted to decimal internally and shifted for a short-word (4 character) print-out while the sixth item was the signed zero indicating presence or absence of alternate parities. (Print-out time for each p was approximately 5 seconds.)

The total running time was about one hour of which 20 minutes was input-output.

The 21 values of $p(<3000)$ for which the A_i have alternate parity (or for which the Θ_i are all positive) are reproduced below with j their order in the list of primes (of the form $6n+1$).

j	p	j	p	j	p
18	163	71	853	143	1879
27	277	77	937	145	1951
33	349	81	1009	156	2131
37	397	107	1399	169	2311
48	547	129	1699	191	2689
52	607	135	1777	199	2797
60	709	137	1789	200	2803

It will be seen that the frequency of these p is close to $\frac{1}{10}$, which is not in agreement with "randomness" of parities (as postulated above). The magnitudes of the A_i did come out, however, to agree rather well with the randomness assumption made earlier, and they are omitted here. A spot check[4] of $p = 163, 277, 349, 2803$ by hand calculations revealed that unique factorization does fail in the field $R(\alpha)$, and that the class number is 4 for each of these cases.

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[1] Compare D. H. LEHMER, *Guide to tables in the theory of numbers*. National Research Council, 1941, p. 75–77, O. TAUSSKY, *Some computational problems in algebraic number theory*. National Bureau of Standards *Report* (to appear).

[2] For theoretical background consult H. HASSE, *Arithmetische Bestimmung von Grundeinheit*. Berlin, 1950, p. 70.

[3] These were taken from a table of minimum positive g for $p < 3000$ in I. M. VINOGRADOV, *Osnovy Teorii Chisel* [*Fundamentals of the Theory of Numbers*]. Moscow, 1940, p. 110.

[4] The case $p = 163$ was discovered through another procedure by E. ARTIN, according to a private communication.

167.—CULLEN NUMBERS. These are numbers of the form $n2^n + 1$ and are remarkable in that they seem to be composite for $n > 1$, although there is no *a priori* reason for this. CUNNINGHAM & WOODALL[1] made a study of these numbers and found them all composite with a small factor for $1 < n < 141$. No factor of $141 \cdot 2^{141} + 1$ is known. I have completely factored the following cases left incomplete by Cunningham. The case $n = 46$ is due to R. A. LIÉNARD of Lyons.

n	$n2^n + 1$	n	$n2^n + 1$
33	$47 \cdot 6031230671$	42	$23 \cdot 43 \cdot 83 \cdot 2250270487$
35	$37 \cdot 32502455213$	43	$3 \cdot 5 \cdot 163 \cdot 2633 \cdot 58752797$
37	$3 \cdot 5 \cdot 339016085231$	45	$11 \cdot 47 \cdot 2437 \cdot 1256655529$
38	$3^2 \cdot 20879 \cdot 55586743$	46	$5 \cdot 31 \cdot 47 \cdot 139297 \cdot 3189821$
39	$41 \cdot 3433 \cdot 152326961$	47	$7 \cdot 11 \cdot 43 \cdot 3593 \cdot 556021079$
40	$41 \cdot 131611 \cdot 8150491$	48	$7 \cdot 379 \cdot 997 \cdot 5107973329$
41	$13 \cdot 43 \cdot 1291 \cdot 124932557$	66	$5^3 \cdot 13 \cdot 67 \cdot 107 \cdot 131 \cdot 8353 \cdot 382030403$

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[1] A. J. C. CUNNINGHAM & H. J. WOODALL, "Factorisation of $Q = (2^q \mp q)$ and $(q \cdot 2^q \mp 1)$," *Messenger Math.*, v. 47, 1917, p. 1–38.

CORRIGENDA

V. 6, p. 225, l. 11, *for* monomial *read* elementary.
V. 7, p. 34, l. 6, *for* 6 *read* 1.
V. 7, p. 175, l. 17, *for* 9 *read* 8.